

# SYMPLECTOMORPHISM GROUP RELATIONS AND DEGENERATIONS OF LANDAU-GINZBURG MODELS

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**ABSTRACT.** In this paper, we describe explicit relations in the symplectomorphism groups of toric hypersurfaces. To define the elements involved, we construct a proper stack of toric hypersurfaces with compactifying boundary representing toric hypersurface degenerations. Our relations arise through the study of the one dimensional strata of this stack. The results are then examined from the perspective of homological mirror symmetry where we view sequences of relations as maximal degenerations of Landau-Ginzburg models. We then study the  $B$ -model mirror to these degenerations, which gives a new mirror symmetry approach to the minimal model program.

## 1. INTRODUCTION

In his 1994 ICM address, Maxim Kontsevich reformulated the concept of mirror symmetry in string theory as a deep mathematical duality now known as homological mirror symmetry. This has united many previously disparate notions in symplectic geometry, algebraic geometry, and category theory. Although originally formulated for Calabi-Yau manifolds, evidence has emerged suggesting that the scope of homological mirror symmetry is much more vast [36].

Developments in last few years have drawn much attention to phase changes in the moduli spaces appearing in mirror symmetry, exemplified in the work on wall-crossing by Gaiotto, Neitzke, Moore [27] and Kontsevich, Soibelman [44], where the moduli spaces involved are the spaces of stability conditions pioneered by Bridgeland [14]. This modern approach to wall-crossing is in many ways a generalization of the geometric invariant theory developed by Mumford, whose variational theory was studied by Dolgachev-Hu [23] and Thaddeus [57]. In another direction, wall-crossing phenomena also generalize the by now classic work of Simpson on the moduli spaces of Higgs bundles [56], as originally conceived by Hitchin [32]. The latter connection suggests the need for a Hodge theoretic approach to stability conditions and wall-crossing, a theory of Stability Hodge Structures [37].

In this paper we concentrate on a piece of this new theory, where one retains explicit combinatorial control. Here the relevant moduli space is a moduli space of Landau-Ginzburg models (LG models), and the mirror symmetry is for toric Deligne-Mumford stacks. We take a minimalist approach to moduli of LG models, developing only aspects which serve the geometric goals of this paper, and defer to the future work [37] for a more systematic treatment. Still, we discover a wide range of unexpected geometric applications. In this paper we observe wall-crossing phenomena manifesting as explicit symplectomorphisms on hypersurfaces.

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We study the geometry of these symplectomorphisms and relations between them in the symplectic mapping class group.

The mapping class group for Riemann surfaces has been studied from a variety of directions and perspectives for many years. Following the ideas of Hatcher, Thurston and others, Wajnryb gave a finite presentation for these groups, thereby opening their structure to computational analysis [59]. The mapping class group of marked curves and curves with boundary was covered in this presentation as well. In all cases, the presentation consists of a generating set of Dehn twists and braids, along with a set of relations obtained from standard relations on model surfaces. For example, the braid, lantern and star relations all occur on punctured genus  $\leq 1$  surfaces and are embedded into the larger surface to yield part of the presentation. Generalizations of these results to diffeomorphism groups in higher dimensions is much less understood. However, considering symplectic manifolds in the context of toric or tropical geometry, the chances of success appear to increase dramatically. This paper aims to supply strong evidence of this.

In this paper, we restrict attention primarily to symplectic orbifold hypersurfaces  $(\mathcal{Y}, \omega)$  contained in a  $d$  dimensional toric stack  $\mathcal{X}_\Sigma$ . Every such hypersurface can be thought of an orbifold with boundary  $\partial\mathcal{Y}$  by considering the intersection with the toric boundary. We then obtain a collection of generators and relations for a subgroup  $\mathbf{G}$  of the symplectic mapping class group  $\pi_0(\mathrm{Symp}(\mathcal{Y}, \partial\mathcal{Y}))$ . Our method is to consider representations of the fundamental group of a stack  $\mathcal{X}^\circ = (\mathcal{X} - \mathcal{E})$  which can be thought of as a toric moduli space for  $(\mathcal{Y}, \omega)$ . As this is a moduli space in the toric category, we avoid many of pathologies regarding classical moduli spaces of hypersurfaces (see Kollár [42]). Here  $\mathcal{X}$  is a proper stack over  $\mathbb{C}$  and the determinant locus,  $\mathcal{E}$ , is a singular, reducible hypersurface which includes the discriminant as a component. The other components of  $\mathcal{E}$  classify hypersurfaces which obtain singularities on their boundary. Removing  $\mathcal{E}$ , we find a symplectic connection on the universal hypersurface  $\mathcal{H}$  over  $\mathcal{X}$  and employ symplectic parallel transport:

$$(1) \quad \mathbf{P} : \Omega_*(\mathcal{X}^\circ) \rightarrow \mathrm{Symp}(\mathcal{Y}, \partial\mathcal{Y})$$

Taking the group  $\mathbf{G} = \pi_0(\mathrm{im}(\mathbf{P}))$ , we find generators and relations by studying them in  $\pi_1(\mathcal{X}^\circ)$ .

Before we can implement this program, we need to define the stacks  $\mathcal{X}$  and  $\mathcal{E}$ . Assume  $\mathcal{Y}$  is obtained as the zero locus of a section of an ample line bundle  $\mathcal{O}_A(1)$  of  $\mathcal{X}_\Sigma$  and  $A \subset \mathbb{Z}^d$  is a subset of the weight polytope of  $H^0(\mathcal{X}_\Sigma, \mathcal{O}(1))$ . Following the techniques of Gelfand, Kapranov, Zelevinsky [46] and Alexeev [4] we construct a toric stack  $\mathcal{X}_{\Sigma(A)}$ , with a stacky fan description, which has an open substack  $\mathcal{V}_A$  corresponding to the moduli of toric hypersurfaces which do not contain any torus fixed point of  $\mathcal{X}_\Sigma$ . The moment polytope for  $\mathcal{X}_{\Sigma(A)}$  is the secondary polytope  $\Sigma(A)$  of  $A$ , but the stack  $\mathcal{X}_{\Sigma(A)}$  is more refined than the variety  $X_{\Sigma(A)}$ .

As mentioned, the substack  $\mathcal{V}_A \subset \mathcal{X}_{\Sigma(A)}$  is a moduli space for certain toric hypersurfaces and the full stack  $\mathcal{X}_{\Sigma(A)}$  is a geometric compactification. More precisely, after Mumford [52], we define a toric hypersurface degeneration and find that all such degenerations are classified by the boundary of  $\mathcal{X}_{\Sigma(A)} - \mathcal{V}_A$ . This is in analogy to the case of Riemann surfaces where we obtain the moduli space as an open subset of the moduli space of stable curves.

To define the universal hypersurface over  $\mathcal{X}_{\Sigma(A)}$ , we first define a toric variety  $\mathcal{X}_{\Theta(A)}$  following the ideas of Lafforgue [45] and regard the universal hypersurface

as a single hypersurface  $\mathcal{Y}_A \subset \mathcal{X}_{\Theta(A)}$ . We show that  $\mathcal{X}_{\Theta(A)}$ , and  $\mathcal{Y}_A$ , have explicit formulations in terms of a toric stack with line bundle, and the zero locus of a universal section, respectively. There is a morphism  $\pi : \mathcal{X}_{\Theta(A)} \rightarrow \mathcal{X}_{\Sigma(A)}$  whose fibers are possibly degenerated toric varieties with a hypersurface. Restricting  $\pi$  to  $\mathcal{Y}_A - \pi^{-1}(\mathcal{E})$  gives a smooth bundle over  $\mathcal{X}^\circ$  where the map in equation 1 can be defined.

In the case of curves, the generators of the mapping class group can be taken to be Dehn twists and braids. One expects a more complicated set of generators to occur in higher dimensions. Indeed, the generators we obtain fall into two different classes, hypersurface degeneration monodromy and stratified Morse function monodromy. The former refers to monodromy around the compactifying divisor  $\partial\mathcal{X}_{\Sigma(A)}$  and has been studied near the points associated to maximal triangulations by Abouzaid [1]. The geometric description of these symplectomorphisms is obtained by first breaking the hypersurface up into its degenerated components where the symplectomorphism is a unitary map, and then convolving along the degenerating vanishing cycle to obtain a global map. For curves, these occur as a combination of a Dehn twist and a finite order map.

The second class of generators we examine are generalizations of spherical twists as investigated by Seidel [54]. Since  $\mathcal{Y}$  has a boundary arising from a normal crossing divisor, a smooth point  $t \in \mathcal{X}$  where the fiber  $\mathcal{Z}$  is smooth, but does not transversely intersect the boundary, also yields a symplectic monodromy map. The local model for monodromy here is a generalization of the usual monodromy around a Morse singularity to that around a stratified Morse singularity as defined in Goresky and Macpherson's work [29]. In order to obtain a well defined symplectic monodromy map in this situation, we must isotope  $\omega$  to a singular symplectic form with singular set along some components of the boundary. This isotopy can be defined on all of  $\mathcal{X}_{\Theta(A)}$ , and conjugating the monodromy map by it, we obtain the modified symplectic parallel transport map. Its description is that of a generalized braid about a Lagrangian submanifold which is a join of a sphere and simplex. This gives twists about Lagrangian discs and balls, as well as other interesting joins.

In general, the fundamental groups of discriminant complements are quite complicated ([22]), to say nothing of determinant complements. In pursuit of relations between the above generators, we follow the example of mapping class groups of curves and find a class of toy models to study. These models occur as subdivided pieces of  $(\mathcal{Y}, \omega)$  for general toric hypersurfaces  $\mathcal{Y}$ . The most elementary piece  $A \subset \mathbb{Z}^d$  occurring in such a subdivision is that of  $(d+1)$  affinely independent lattice points whose convex hull is a simplex. The corresponding toric hypersurface is a higher dimensional pair of pants (see [51]). However, simplices have secondary stacks  $\mathcal{X}_{\Sigma(A)} = BG$  where  $G$  is a finite abelian group and so their fundamental group does not contain any interesting relations. The next level of complexity is that of an affine circuit which generically is a set of  $(d+2)$  points in  $\mathbb{Z}^d$  that span  $\mathbb{R}^d$ . In this setting we already observe a rich interplay between spherical twists, degeneration monodromy and finite group actions. The tropical picture illustrates these circuits as Morse functions related to bistellar flips. For the most elementary cases in dimension 1, we recover the lantern and star relations.

More generally, we obtain the following abridged version of one of our main theorems in section 4.1.

**Theorem 1.1.** *Let  $A$  be a circuit affinely spanning  $\mathbb{Z}^d$ . There are symplectomorphisms  $T_0, T_1, T_\infty \in \text{Symp}(\mathcal{Y}, \partial\mathcal{Y})$  with  $T_0$  and  $T_\infty$  toric degeneration monodromy maps and  $T_1$  the monodromy about a stratified Morse singularity. In the mapping class group  $\pi_0(\text{Symp}(\mathcal{Y}, \partial\mathcal{Y}))$ , these satisfy the relation:*

$$(2) \quad T_0 T_1 T_\infty = \tau(\mathbf{t})$$

where  $\tau(\mathbf{t})$  is a rotation about the boundary  $\partial\mathcal{Y}$ .

In order to put the generators and relations into a symplectomorphism group of a general  $(\mathcal{Y}, \omega)$ , we address the process of regeneration of circuits. This allows us to import relations obtained over the boundary strata of  $\mathcal{X}_{\Sigma(A)}$  into the interior. As explained in [7], the one dimensional boundary strata of  $\mathcal{X}_{\Sigma(A)}$  correspond to circuits supported on  $A$ . After regenerating these circuits, we obtain a host of geometrically meaningful relations between generators in  $\mathbf{G}$ . Taking a general map  $\phi : \mathbb{P}^1 \rightarrow \mathcal{X}_{\Sigma(A)}$  and pulling back the universal hypersurface gives a framed Lefschetz pencil over  $\mathbb{P}^1$ . We describe a presentation of the monodromy group associated to such pencils by isotoping them near the boundary of  $\mathcal{X}_{\Sigma(A)}$  and relating the bubbled components to circuits. This gives a combinatorial description not only of the groups involved, but their action on the hypersurface.

Guided by mirror symmetry we view the Lefschetz pencil as a Landau-Ginzburg, or LG, model. This topic has its roots in quantum field theory and has found many mathematical applications via mirror symmetry [33]. Our perspective takes a LG model as a curve  $i : \mathcal{C} \rightarrow \mathcal{X}_{\Sigma(A)}$ . We pay special attention to curves that are obtained as compactifications of one parameter orbits in  $\mathcal{X}_{\Sigma(A)}$  which we call sharpened pencils. All examples of mirrors of Fano toric stacks lie in this class [33]. Following results of [47], we observe that the coarse moduli space of these LG models has a natural compactification as a toric variety whose moment polytope is the monotone path polytope of  $\Sigma(A)$  as investigated in [8], [9]. By analogy to the case of  $\mathcal{X}_{\Sigma(A)}$  where the fixed points correspond to large volume complex structure limits, the fixed points of this toric variety correspond to maximal degenerations of the LG model. Every such fixed point is related to a vertex of the monotone path polytope which itself gives an edge path on  $\Sigma(A)$ . As mentioned above, this edge path gives a sequence of circuits describing the original Lefschetz pencil. One main application of our work is to use any such sequence to describe an associated semi-orthogonal decomposition of the Fukaya-Seidel category of the LG model.

This semi-orthogonal decomposition complements recent developments in homological mirror symmetry for the case of Fano varieties. Work of Bondal-Orlov [11] and Kawamata [39] illustrates relations between birational transformations and semi-orthogonal decompositions; see also [6] for a thorough treatment of this phenomenon for birational maps obtained from Geometric Invariant Theory. One thus expects a mirror sequence of birational maps corresponding to the degenerated Landau-Ginzburg model. In the toric case, the equivariant birational geometry is well-understood combinatorially and goes back to the work of Reid [53]. The fact that these birational transformations are typically centered on the toric boundary of the toric variety means that we are working with log pairs, see the paper of Kollár [41] for foundations on the birational geometry of log pairs.

We show concretely that degenerations of toric Landau-Ginzburg models correspond bijectively to possible runs of the minimal model program for toric varieties. As a consequence we obtain a concise description of the mirrors of toric flips and

toric divisorial contractions in terms of circuits. We conjecture that there is an equivalence of categories which restricts to this identification of semiorthogonal components, giving a clear picture of the geometry underlying homological mirror symmetry for toric DM stacks. We give evidence for this conjecture by computing ranks in  $K$ -theory, along the way extending results of Borisov-Horja [13]. We note that it is crucial to allow for toric stacks in the above minimal model process; this is consistent with Kawamata's results [39]. Moreover, our work implies that one dimensional strata on the bottom of the monotone path polytope correspond to Mori fiber spaces, two dimensional strata to Sarkisov links, three dimensional strata to relations between Sarkisov links, and so on.

The relationship between the minimal model sequences of  $\mathcal{X}_\Sigma$  and the mirror A-model LG degenerations on  $\mathcal{X}_\Sigma^{mir}$  is suggested by the theorem 5.14, which in simple cases reduces to the following statement.

**Theorem 1.2.** *The set of minimal model sequences of  $\mathcal{X}_\Sigma$  for a Fano toric stack are in bijective correspondence with the set of mirror sequences to maximal degenerations of the superpotential on the mirror stack  $\mathcal{X}_\Sigma^{mir}$ . Both are in bijective correspondence with vertices of a monotone path polytope  $\Sigma_\rho(\Sigma(A))$ .*

Although our main application concerns mirrors of toric stacks, it is worthwhile to view these results in light of recent developments in the minimal model program for more general varieties. We refer to the classic text of Kollár and Mori [43] for foundational aspects. The works of Birkar, Cascini, Hacon, McKernan [10] and Corti, Lazic [17] have drawn attention to certain loci in the pseudoeffective cone of a variety, called Shokurov polytopes, where finite generation of the corresponding log canonical rings holds, and there is a finite chamber decomposition coming from distinct birational models. In the toric case the effective cone is itself polyhedral, and the chamber decomposition is well studied [19], and coincides with the mirror correspondence discussed above. We expect that, beyond the toric case, the boundary of the moduli space of LG models mirror to a Fano variety  $X$  sheds light on the chamber decomposition of the pseudoeffective cone of  $X$ . Beyond the Fano case, the techniques of this paper can also be used to generate Shokurov polytopes when a LG mirror is known, even if the pseudoeffective cone is itself not polyhedral.

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## 2. TORIC PRELIMINARIES

In this section, we will give key definitions and constructions for the toric moduli space of hypersurfaces and its compactification. An important point to keep in mind throughout is that our moduli stacks are only of toric hypersurfaces, and only up to toric isomorphism, not general isomorphisms. The advantage of this approach is that we obtain stacks with extremely explicit representations.

In the first two subsections we recall and collect notions of the algebraic and symplectic geometry of toric stacks. Many familiar aspects of this subject will be assumed, but all novel constructions will be discussed. In the last two subsections, we recall the constructions of Gelfand, Kapranov and Zelevinsky [46] and Lafforgue

[45]. We adapt these ideas into the definition of several toric stacks which give the moduli compactification, a universal toric variety lying over it and its universal hypersurface.

**2.1. Toric preliminaries.** We start this section by recalling the construction of toric stacks through the data of a stacky fan. We utilize the material in [28] rather than the more classical approach given in [18], [12]. This allows us to incorporate more general Artin stacks into the discussion.

**Definition 2.1.** A stacky fan  $\Sigma$  consists of the data  $(\Lambda_1, \Lambda_2, \beta, \Sigma)$  where:

- (i)  $\Lambda_2$  is a finitely generated abelian group,
- (ii)  $\Sigma$  is a fan in  $\Lambda_1 \otimes \mathbb{R}$  and  $\Lambda_1$  is a lattice,
- (iii)  $\beta : \Lambda_1 \rightarrow \Lambda_2$  is a homomorphism with finite cokernel.

We call  $\Sigma$  good if the  $\Lambda_1$  primitives  $\Sigma(1)$  are linearly independent and span a saturated sublattice of  $\Lambda_2$ .

For the rest of the paper, we will identify  $\Sigma(1)$  with its primitive generators. Note that our definition of a stacky fan corresponds to a generically stacky fan with close  $\beta$  in [28]. The notion of a good stacky fan will be explained after we recall the stack associated to  $\Sigma$ . First we extend  $\beta$  to an exact sequence

$$(3) \quad 0 \rightarrow L_\Sigma \xrightarrow{\alpha} \Lambda_1 \xrightarrow{\beta} \Lambda_2 \rightarrow K_\Sigma \rightarrow 0.$$

After tensoring with  $\mathbb{C}^*$  we have that the kernel of  $(\beta \otimes_{\mathbb{Z}} 1)$  is

$$(4) \quad \mathbb{H}_\Sigma := (L_\Sigma \otimes \mathbb{C}^*) \oplus \text{Tor}(K_\Sigma, \mathbb{C}^*).$$

This leads to the definition:

**Definition 2.2.** Given a stacky fan  $\Sigma$ , the toric stack  $\mathcal{X}_\Sigma$  is defined to be the quotient stack  $\mathcal{X}_\Sigma = [X_\Sigma / \mathbb{G}_\Sigma]$ . If  $\Sigma$  is good, we call  $\mathcal{X}_\Sigma$  a good toric stack.

Note that for any  $\lambda \in \mathbb{G}_\Sigma := \Lambda_2 \otimes \mathbb{C}^*$ , we may choose  $\lambda' \in \Lambda_1 \otimes \mathbb{C}^*$  with  $\beta(\lambda') = \lambda$  and define  $\lambda \cdot - : \mathcal{X}_\Sigma \rightarrow \mathcal{X}_\Sigma$  by  $\lambda' \cdot \mathbf{z}$  for  $\mathbf{z} \in X_\Sigma$ . This defines the torus action of  $\mathbb{G}_\Sigma$  on  $\mathcal{X}_\Sigma$  up to natural isomorphisms. The action can be made strict in the cases that  $K_\Sigma$  is trivial.

Given two stacky fans,  $\tilde{\Sigma}$  and  $\Sigma$ , we define a map  $g : \tilde{\Sigma} \rightarrow \Sigma$  to be a pair  $(g_1, g_2)$  such that  $g_1 : \tilde{\Lambda}_1 \rightarrow \Lambda_1$  induces a map of fans  $g_1 : \tilde{\Sigma} \rightarrow \Sigma$  and  $g_2 : \tilde{\Lambda}_2 \rightarrow \Lambda_2$  satisfies  $\beta \circ g_1 = g_2 \circ \tilde{\beta}$ . It is clear that any such map of stacky fans induces a map  $\tilde{g} : \mathcal{X}_{\tilde{\Sigma}} \rightarrow \mathcal{X}_\Sigma$ . If  $\tilde{\lambda} \in \mathbb{G}_{\tilde{\Sigma}}$  and  $\lambda \in \mathbb{G}_\Sigma$  we write  $\tilde{g}_{\tilde{\lambda}, \lambda} := \lambda \cdot \tilde{g}(\tilde{\lambda} \cdot z)$ .

It was shown that in [28] that any toric stack  $\mathcal{X}$  has a canonical stack  $\mathcal{X}_\Sigma \rightarrow \mathcal{X}$  where  $\Sigma$  is a good stacky fan. This map satisfies a universal property and can be thought of as a stacky resolution of  $\mathcal{X}$ . When  $\mathcal{X} = \mathcal{X}_\Sigma$  and  $\tilde{\Sigma}$  is good, the map is an isomorphism. All of the toric stacks defined and worked with in this paper will be good and most will be Deligne-Mumford stacks.

We recall from [18] that, for a good DM toric stack  $\mathcal{X}_\Sigma$ , one can identify the space of equivariant Cartier divisors  $\text{Div}_{eq}(\mathcal{X}_\Sigma)$  with  $\Lambda_1^\vee$  and the Picard group with  $\text{Pic}(\mathcal{X}_\Sigma) = L_\Sigma^\vee \oplus \text{Ext}^1(K_\Sigma, \mathbb{Z})$ . Indeed, letting  $\Sigma^\vee \subset \Lambda_1^\vee$  be the dual cone to the cone over  $\Sigma(1)$ , the ring  $R_\Sigma = \mathbb{C}[x_\sigma : \sigma \in \Sigma^\vee(1)]$  is the homogeneous coordinate ring for  $X_\Sigma$  graded by the character lattice  $L_\Sigma^\vee \oplus \text{Ext}^1(K_\Sigma, \mathbb{Z})$  of  $\mathbb{G}_\Sigma$ . Given  $\gamma_0 \in \Lambda_1^\vee$ , we write  $D_{\gamma_0}$  for the associated Cartier divisor and  $\mathcal{O}(D_{\gamma_0})$  for the line bundle in  $\text{Pic}(\mathcal{X}_\Sigma)$ . For any character  $\gamma_0 \in \Sigma^\vee$  the set

$$(5) \quad [\gamma_0] = \{\gamma \in \Sigma^\vee : \alpha_\Sigma^\vee(\gamma) = \alpha_\Sigma^\vee(\gamma_0)\} \subset \Lambda_1^\vee$$

gives the space  $H^0(\mathcal{X}_\Sigma, \mathcal{O}(D_{\gamma_0}))$  an eigenvector decomposition into  $(\mathbb{C}^{[\gamma_0]})^\vee = \text{Hom}_{\text{set}}([\gamma_0], \mathbb{C})$  with eigenbasis consisting of the monomials  $\{x_\gamma : \gamma \in [\gamma_0]\} \subset R_\Sigma$ . When the divisor  $\gamma$  is chosen, the group  $\mathbb{G}_\Sigma \times \mathbb{C}^*$  acts on  $H^0(\mathcal{X}_\Sigma, \mathcal{O}(D_\gamma))$  via

$$(6) \quad (\lambda, t) \left( \sum_{\gamma \in [\gamma_0]} c_\gamma x_\gamma \right) = t \sum_{\gamma \in [\gamma_0]} (\beta^\vee)^{-1}(\gamma - \gamma_0)(\lambda) c_\gamma x_\gamma.$$

Where we have identified  $\Lambda_2^\vee$  with the group of characters  $\text{Hom}(\mathbb{H}_\Sigma, \mathbb{C}^*)$ .

Suppose  $g : \tilde{\Sigma} \rightarrow \Sigma$  is a map of stacky fans and  $\gamma \in \Lambda_1^\vee \cap \Sigma^\vee$  a positive divisor on  $\mathcal{X}_\Sigma$ , then the map

$$(7) \quad \tilde{g}^* : H^0(\mathcal{X}_\Sigma, \mathcal{O}(D_\gamma)) \rightarrow H^0(\mathcal{X}_{\tilde{\Sigma}}, \mathcal{O}(D_{g_1^\vee(\gamma)}))$$

is simply

$$(8) \quad \tilde{g}^* \left( \sum_{\gamma \in [\gamma_0]} c_\gamma x_\gamma \right) = \sum_{\gamma \in [\gamma_0]} c_\gamma x_{g_1^\vee(\gamma)}.$$

Now assume that  $g : \tilde{\Sigma} \rightarrow \Sigma$  describes a flat morphism of good toric stacks. Recall from [35] that such a map has the property that  $g_1$  maps  $\tilde{\Sigma}(1)$  onto  $\Sigma(1)$  implying that  $g_1 : \tilde{\Lambda}_1 \rightarrow \Lambda_1$  has cofinite image  $\Gamma_1 := \text{im}(g_1)$ . Let  $\Gamma_2$  be the pushout

$$(9) \quad \begin{array}{ccc} \tilde{\Lambda}_1 & \xrightarrow{\tilde{\beta}} & \tilde{\Lambda}_2 \\ g_1 \downarrow & & h \downarrow \\ \Gamma_1 & \xrightarrow{\gamma} & \Gamma_2 \end{array}$$

and define:

**Definition 2.3.** Given an equivariant flat morphism  $g : \mathcal{X}_{\tilde{\Sigma}} \rightarrow \mathcal{X}_\Sigma$  between two good toric stacks, let:

- (i)  $\Sigma_g^f = (\tilde{\Lambda}_1, \Gamma_2, \tilde{\Sigma}, \gamma \circ g_1)$  and  $\mathcal{X}_g^f = \mathcal{X}_{\Sigma_g^f}$  be called the fine quotient relative to  $g$ ,
- (ii)  $\Sigma_g^\rightarrow = (\Gamma_1, \Gamma_2, \Sigma, \gamma)$  and  $\mathcal{X}_g^\rightarrow = \mathcal{X}_{\Sigma_g^\rightarrow}$  be called the colimit stack relative to  $g$ , and  $g^\rightarrow = (g_1, h) : \mathcal{X}_{\tilde{\Sigma}} \rightarrow \mathcal{X}_g^\rightarrow$  the induced morphism.

Note that both constructions yield good toric stacks.

**Proposition 2.4.** Suppose  $\mathcal{X}_1$ ,  $\mathcal{X}_2$  and  $\mathcal{X}_3$  are good toric stacks. Let  $g : \mathcal{X}_1 \rightarrow \mathcal{X}_3$ ,  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  and  $h : \mathcal{X}_2 \rightarrow \mathcal{X}_3$  be equivariant morphisms with  $g = h \circ f$  flat. If  $h$  is a bijection on orbits then there exists a unique map  $\tilde{h} : \mathcal{X}_g^\rightarrow \rightarrow \mathcal{X}$  such that  $f = \tilde{h} \circ g^\rightarrow$ .

The proof of this follows from the universal properties of pushout along with the assumption that  $h$  is an isomorphism on fans defining  $\mathcal{X}_2$  and  $\mathcal{X}_3$ . The motivation for the proposition is to show that  $\mathcal{X}_g^\rightarrow$  is the finest toric stack which lies over  $\mathcal{X}_3$  and under  $\mathcal{X}_1$ . This fact will be used to obtain the ideal moduli space satisfying such a condition.

Many of the toric stacks we work with arise from considering finite sets in a lattice or finitely generated abelian group. It will therefore be helpful to have a standard notation for the exact sequence arising in these situations. Let  $B$  be a finite subset of a finitely generated abelian group  $\Lambda$ . Let  $\beta_B : \mathbb{Z}^B \rightarrow \Lambda$  be the

homomorphism given by assigning  $e_b$  to  $b$  where  $\{e_b : b \in B\}$  is the standard basis for  $\mathbb{Z}^B$ . We call the exact sequence

$$(10) \quad 0 \rightarrow L_B \xrightarrow{\alpha_B} \mathbb{Z}^B \xrightarrow{\beta_B} \Lambda \rightarrow K_B \rightarrow 0$$

the fundamental sequence associated to  $B$ . We also use the following notation for the derived dual

$$(11) \quad 0 \rightarrow \Lambda^\vee \xrightarrow{\beta_B^\vee} (\mathbb{Z}^B)^\vee \xrightarrow{\alpha_B^\star} L_B^\vee \oplus \text{Ext}^1(K_B, \mathbb{Z}) \rightarrow 0,$$

where  $\alpha_B^\star = \alpha^\vee \oplus \delta$  if  $\delta$  is the connecting homomorphism for the long exact Ext sequence. We will also use the notation  $\Lambda_{B^\vee}$  for  $L_B^\vee \oplus \text{Ext}^1(K_B, \mathbb{Z})$ .

In many cases, we will utilize the fundamental sequence when defining a stacky fan, but there are situations where we will need only the homomorphisms and the associated groups. When  $B$  comes equipped with an abstract simplicial complex  $\mathcal{B} \subset \mathcal{P}(B)$ , one defines the fan  $\Sigma_B$  in  $\mathbb{R}^B$  as containing cones  $\text{Cone}(\tau) = \text{Span}_{\mathbb{R}_{\geq 0}} \{e_b : b \in \tau\}$  for every  $\tau \in \mathcal{B}$ . In this situation, we write  $\Sigma_{B, \mathcal{B}} = (\mathbb{Z}^B, \Gamma, \beta_B, \Sigma_B)$  and  $\mathcal{X}_{B, \mathcal{B}}$  for the associated stack. If  $\mathcal{B}$  is understood, we may write  $\Sigma_B$  and  $\mathcal{X}_B$ . Note that all stacky fans in the sense of [12] and fantastacks from [28] are obtained from this construction.

Suppose  $\Lambda$  is a rank  $d$  lattice. Let  $A \subset \Lambda$  be a subset which affinely spans  $\Lambda \otimes \mathbb{R}$  and  $Q = \text{Conv}(A) \subset \Lambda_{\mathbb{R}}$ . By a marked polyhedron we mean a pair  $(Q, A)$  where  $Q$  is a polyhedron, i.e. the intersection of finitely many half spaces in  $\Lambda \otimes \mathbb{R}$ . In this case, we take  $\overline{Q} \subset \Lambda^\vee$  to be the finite set of primitive generators for supporting hyperplanes of  $Q$ . More precisely, for every  $b \in \Lambda^\vee$  let  $n_b = -\min\{b(v) : v \in Q\}$ . Then  $b \in \overline{Q}$  if and only if  $b$  is primitive and  $\{v \in Q : b(v) = n_b\}$  is facet of  $Q$ . We note that the dual of the face poset of  $Q$  then defines an abstract simplicial complex  $\mathcal{Q}$  on  $\overline{Q}$ . The additional data of  $(Q, A)$  yields a positive line bundle  $\mathcal{O}(D_{\gamma_A})$  where  $\gamma_A = \sum_{b \in \overline{Q}} n_b e_b^\vee \in (\mathbb{Z}^{\overline{Q}})^\vee$  and a linear system  $(\mathbb{C}^A)^\vee \subset (\mathbb{C}^{[\gamma]})^\vee = H^0(\mathcal{X}_{\overline{Q}, \mathcal{Q}}, \mathcal{O}(D_\gamma))$ .

**Definition 2.5.** Given a marked polyhedron  $(Q, A)$ , we say the  $\mathcal{X}_Q := \mathcal{X}_{\overline{Q}, \mathcal{Q}}$  is the toric stack associated to  $A$  with boundary divisor  $\partial \mathcal{X}_Q = D_{\sum_{b \in \overline{Q}} e_b^\vee}$ , line bundle  $\mathcal{O}_A(1) := \mathcal{O}(D_{\gamma_A})$  and linear system  $\mathcal{L}_A \subset H^0(\mathcal{X}_Q, \mathcal{O}_A(1))$ .

It is not hard to detail the correspondence of torus orbits to faces to the stacky case for a marked polyhedron  $(Q, A)$ . For any face  $Q'$ , we will write  $\text{orb}_{Q'} \subset \mathcal{X}_Q$  for the corresponding orbit.

The study of toric varieties and stacks from the perspective of marked polytopes places the linear system as a central object of study. Those sections that have singularities on various orbits of  $\mathcal{X}_Q$  will be of particular interest. Let  $A_v$  be the set of vertices of  $Q$  and  $A_{nv} = A - A_v$ .

**Definition 2.6.** A section  $s \in \mathcal{L}_A \subset H^0(\mathcal{X}_Q, \mathcal{O}_A(1))$  is degenerate if  $Y_s \cap \text{orb}_{Q'}$  is singular for some face  $Q'$  of  $Q$ . If  $s = \sum_{\alpha \in A} c_\alpha x_\alpha$  we say  $s$  is full if  $c_\alpha \neq 0$  for all  $\alpha \in A_v$  and very full if  $c_\alpha \neq 0$  for all  $\alpha \in A$ .

When  $\mathcal{X}_Q$  is a smooth stack, a degenerate section is a section which does not transversely intersect the toric boundary. The principal determinant,

$$(12) \quad E_A : \mathcal{L}_A \rightarrow \mathbb{C}$$



is a polynomial which vanishes on degenerate sections. We also recall that the discriminant  $\Delta_A : \mathcal{L}_A \rightarrow \mathbb{C}$  is a polynomial that vanishes on the closure of the set of sections with a singularity in the maximal torus orbit of  $\mathcal{X}_Q$ . It is worth mentioning that there exist sets  $A$  for which the discriminant  $\Delta_A$  is a constant. These cases yield toric varieties that are called dual defect and are studied in [20].

Our next aim is to review the procedure of equipping  $\mathcal{X}_Q$  with an invariant symplectic structure. We will follow the usual route of symplectic reduction [5]. We take  $\mathbb{T} = \{z \in \mathbb{C}^* : |z| = 1\}$  and, given any lattice  $\Gamma$ , we write  $\mathbb{T}_\Gamma$  and  $\mathfrak{t}_\Gamma \approx \Gamma_\mathbb{R}$  for the real torus  $\mathbb{T} \otimes \Gamma$  and its Lie algebra. Recall the fundamental sequence

$$(13) \quad 0 \rightarrow L_{\overline{Q}} \xrightarrow{\alpha_{\overline{Q}}} \mathbb{Z}_{\overline{Q}} \xrightarrow{\beta_{\overline{Q}}} \Lambda^\vee \rightarrow K_{\overline{Q}} \rightarrow 0.$$

We note that the toric variety  $X_{\Sigma_{\overline{Q}, \mathcal{Q}}} \subset \mathbb{C}_{\overline{Q}}$  is an open equivariant subset, so that restricting the standard Kähler structure on  $\mathbb{C}_{\overline{Q}}$  to  $X_{\Sigma_A}$  yields the moment map  $\mu_{\overline{Q}} : X_{\Sigma_A} \rightarrow \mathbb{R}_{\geq 0}^{\overline{Q}}$  given by

$$(14) \quad \mu_{\overline{Q}}(z_1, \dots, z_{|\overline{Q}|}) = (|z_1|^2, \dots, |z_{|\overline{Q}|}|^2),$$

where we have chosen the action of  $\mathbb{T}_{\mathbb{Z}_{\overline{Q}}}$  on  $\mathbb{C}_{\overline{Q}}$  to be

$$(15) \quad (\theta_1, \dots, \theta_{|\overline{Q}|}) \cdot (z_1, \dots, z_n) = (e^{-2i\theta_1} z_1, \dots, e^{-2i\theta_{|\overline{Q}|}} z_{|\overline{Q}|}).$$

On the other hand, restricting to the  $\mathbb{T}_{L_{\overline{Q}}}$  action gives the moment map  $\mu_{L_{\overline{Q}}} = \mu_{\overline{Q}} \circ \alpha_{\overline{Q}}^\vee$  where  $\alpha_{\overline{Q}}^\vee : \mathfrak{t}_{\mathbb{Z}_{\overline{Q}}}^\vee \rightarrow \mathfrak{t}_{L_{\overline{Q}}}^\vee$  is just tensoring with  $\mathbb{R}$  and taking the dual. Choosing a value  $\omega$  in the interior of the image of  $\mu_{L_{\overline{Q}}}$  gives a symplectic form on  $\mathcal{X}_Q$  via the symplectic reduction

$$(16) \quad (\mathcal{X}_Q, \omega) = [\mu_{L_{\overline{Q}}}^{-1}(\omega) / \mathbb{T}_{L_{\overline{Q}}}] .$$

If no choice of  $\omega$  is mentioned, we set  $\omega = \alpha^\vee(D_\gamma)$  and call this the standard symplectic form on  $\mathcal{X}_Q$ . Such a choice fixes  $\mathcal{X}_Q$  as a monotone symplectic stack, which can be thought of as a very stringent condition [49]. After having chosen a symplectic form on  $\mathcal{X}_Q$ , we recover the moment map of  $\mathbb{T}_{\Lambda^\vee}$  on  $\mathcal{X}_Q$  by first considering  $\tilde{\Lambda}^\vee = \beta(\Lambda)$  and the moment map with respect to  $\mathbb{T}_{\tilde{\Lambda}^\vee}$ . We have that, for this group, there is a splitting  $i : \mathbb{T}_{\tilde{\Lambda}^\vee} \rightarrow \mathbb{T}_{\mathbb{Z}_{\overline{Q}}}$  of  $\beta$ . From the exactness of the modified sequence 13

$$(17) \quad 0 \rightarrow L_{\overline{Q}} \xrightarrow{\alpha_{\overline{Q}}} \mathbb{Z}_{\overline{Q}} \xrightarrow{\beta_{\overline{Q}}} \tilde{\Lambda}^\vee \rightarrow 0,$$

we have that  $\tilde{\mu}_A : \mathcal{X}_Q \rightarrow \mathfrak{t}_{\tilde{\Lambda}^\vee} \approx \tilde{\Lambda}_\mathbb{R}$  is given by  $i^* \circ \mu_{\overline{Q}}$ . To recover the actual moment map, we need only compose with the natural map  $\tilde{\Lambda}_\mathbb{R} \rightarrow \Lambda_\mathbb{R}$  inverse to the dual of the inclusion. These moment maps fit into the commutative diagram

$$(18) \quad \begin{array}{ccc} \mu_{L_{\overline{Q}}}^{-1}(\omega) & \xrightarrow{\rho_A} & \mathcal{X}_Q \\ \downarrow \mu_{\overline{Q}} & & \downarrow \mu_A \\ \mathbb{R}_{\overline{Q}} & \xleftarrow{\tilde{\beta}_Q^\vee} & \Lambda_\mathbb{R} \end{array}$$

where  $\tilde{\beta}_Q^\vee = \beta_Q^\vee + \gamma$  for some  $\gamma \in \mathbb{R}_{\overline{Q}}$  with  $\alpha^\vee(\gamma) = \omega$  (note that a different choice will simply translate the moment map).

We observe that the image of the moment map on  $\mathcal{X}_Q$  can be seen as the intersection of an affine subspace  $i(\Lambda_{\mathbb{R}}) + \omega'$  with the positive cone  $\mathbb{R}_{\geq 0}^{\bar{Q}}$ . For the case of the standard form, we observe that the image of  $\mu_A$  is  $Q$  itself. This can be seen by utilizing  $\gamma_A$  in the above definition of  $\tilde{\beta}_{\bar{Q}}$ .

**2.2. Toric hypersurface degenerations.** We now review the procedure for simultaneous degeneration of a toric stack and its hypersurface (see [52], [30]). Recall that  $A \subset \Lambda$  gives a subset of equivariant linear sections of a line bundle  $\mathcal{O}_A(1)$  on a toric stack  $\mathcal{X}_Q$  specified by the stacky fan  $\Sigma_{\bar{Q}, Q}$ . Given a section  $s \in \mathbb{C}^A \subset H^0(\mathcal{X}_Q, \mathcal{O}_A(1))$  in the linear system defined by  $A$ , write  $\mathcal{Y}_s$  for its zero locus and call the pair  $(\mathcal{X}_Q, \mathcal{Y}_s)$  a toric hypersurface. Two such pairs,  $(\mathcal{X}_Q, \mathcal{Y}_s)$  and  $(\mathcal{X}'_Q, \mathcal{Y}_{s'})$  will be considered equivalent if there exists an equivariant isomorphism from  $\mathcal{X}_Q$  to  $\mathcal{X}'_Q$  which pulls back  $s'$  to  $s$ .

We recall the definition of a regular marked subdivision  $S = \{(Q_i, A_i)\}_{i \in I}$  of  $(Q, A)$  from [46]. For each  $i \in I$ ,  $A_i \subset A$  and  $Q_i = \text{Conv}(A_i)$ , the union of the  $Q_i$  is  $Q$ , and the intersection of any two  $Q_i$  is a face of each. Note that the union of all  $A_i$  is not necessarily the set  $A$ . The added condition of regularity is formulated in the following way. Let  $\eta : A \rightarrow \mathbb{R}$  be any function and take

$$(19) \quad Q_\eta = \text{Conv}\{(\alpha, t) \in \Lambda_{\mathbb{R}} \oplus \mathbb{R} : \alpha \in A, t \geq \eta(\alpha)\}$$

to be the convex hull of the half lines defined by  $\eta$ . Let  $\tilde{\eta} : Q \rightarrow \mathbb{R}$  be the function

$$(20) \quad \tilde{\eta}(r) = \min\{t : (q, t) \in Q_\eta\}.$$

By definition,  $\tilde{\eta}$  is a convex, piecewise affine function on  $Q$ . We say that  $\eta$  defines the subdivision  $S = \{(Q_i, A_i) : i \in I\}$  if

- 1)  $\tilde{\eta}|_{Q_i}$  extends to an affine function  $\varsigma_i$  on  $\Lambda \otimes \mathbb{R}$ ,
- 2)  $\eta(\alpha) = \varsigma_i(\alpha)$  if and only if  $\alpha \in A_i$ .

An example of a the graph  $\{(\alpha, \eta(\alpha)) : \alpha \in A\}$  of the function  $\eta$  and its associated polyhedron  $Q_\eta$  is illustrated in figure 2.2.

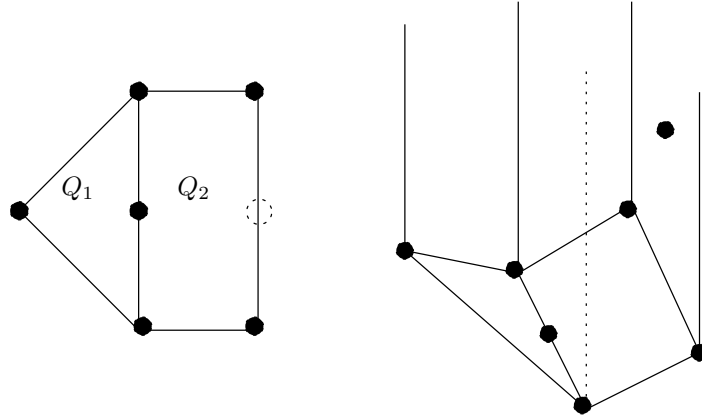


FIGURE 1.  $S = \{(Q_1, A_1), (Q_2, A_2)\}$  and a defining function  $\eta$

For any regular subdivision  $S$ , we take  $C_{\mathbb{R}}^{\circ}(S)$  to be the cone of all defining functions for  $S$  and  $C_{\mathbb{Z}}^{\circ}(S) = (\mathbb{Z}^A)^{\vee} \cap C_{\mathbb{R}}^{\circ}(S)$  the set of integral defining functions.

Write  $C_{\mathbb{R}}(S)$  for its closure and  $C_{\mathbb{Z}}(S) = (\mathbb{Z}^A)^{\vee} \cap C_{\mathbb{R}}(S)$ . For any  $\eta \in C_{\mathbb{Z}}^{\circ}(S)$ , we define

$$(21) \quad A_{\eta} = \{(r, t) \in \Lambda \oplus \mathbb{Z} : r \in A, t \leq \eta(r)\}$$

and write  $(Q_{\eta,i}, A_{\eta,i})$  for the marked facet of  $(Q_{\eta}, A_{\eta})$  over  $Q_i$ .

We will now use integral defining functions to construct and study a degeneration stack of  $\mathcal{X}_Q$ . This technique follows that of Mumford in [52]. Let  $\eta \in C_{\mathbb{Z}}^{\circ}(S)$  and write  $\mathcal{X}_{\eta}$  for the toric stack  $\mathcal{X}_{A_{\eta}}$  as constructed in 2.1. Recall that  $\overline{Q}_{\eta}$  is in bijection with the facets of the polyhedron  $Q_{\eta}$ . Then  $\overline{Q}_{\eta}$  can be written as the disjoint union  $\overline{Q}_{\eta}^v \cup \overline{Q}_{\eta}^h$  of two types of facets where  $v$  and  $h$  refer to vertical and horizontal divisors. The first type,  $b \in \overline{Q}_{\eta}^v$ , consists of the lower boundary which are in one to one correspondence with all of the subdivided pairs  $(Q_i, A_i)$  of  $S$ . The second type of facet in  $\overline{Q}_{\eta}^h$  are the facets of  $Q_{\eta}$  which are in one to one correspondence with the facets of  $Q$  itself. As always, the abstract simplicial complex  $\mathcal{Q}_{\eta}$  on  $\overline{Q}$  is dual to the face poset of  $Q_{\eta}$ .

We notice that the combinatorics of the polyhedron  $Q_{\eta}$  and thus those of  $\mathcal{Q}_{\eta}$  and  $\Sigma_{\overline{Q}_{\eta}, \mathcal{Q}_{\eta}}$  are dictated by  $S$  and not  $\eta$ . The role that  $\eta$  plays in the definition of  $\mathcal{X}_{\eta}$  is in the function  $\beta_{\overline{Q}_{\eta}} : \mathbb{Z}^{\overline{Q}_{\eta}} \rightarrow (\Lambda \oplus \mathbb{Z})^{\vee}$ . The sub-fan  $\Sigma_{A_{\eta}}$  consisting of one cones in  $\overline{Q}_{\eta}^v$  projects via  $\beta_{\overline{Q}_{\eta}}$  to the discrete Legendre transform of  $\eta$  yielding the fan  $\beta_{\overline{Q}_{\eta}}(\Sigma_{A_{\eta}}) \subset (\Lambda_{\mathbb{R}} \oplus \mathbb{R})^{\vee}$  with 1-cones given by  $\beta_{\overline{Q}_{\eta}}(\overline{Q}_{\eta}^v) = \{f - d_{\zeta_i} : i \in I\}$  where  $f = (0, 1) \in (\Lambda \oplus \mathbb{Z})^{\vee}$ . A subtle point about this formula is that, when  $A_i$  affinely spans a positive index sublattice of  $\Lambda$ , the element  $f - d_{\zeta_i}$  is not necessarily in  $(\Lambda \oplus \mathbb{Z})^{\vee}$ . In this case, it is necessary to take a multiple to obtain a primitive generator. We write  $c_{\eta,i} \in \mathbb{Z}_{>0}$  for the denominator of  $d_{\zeta_i}$ . It is not hard to see that  $c_{\eta,i}[\text{Aff}_{\mathbb{Z}}(A_i) : \Lambda]$ , so in general there are only a finite number of possible constants.

As always, the stack  $\mathcal{X}_{\eta}$  comes equipped with a line bundle  $\mathcal{O}_{\eta}(1)$  such that the vector space  $\mathbb{C}^{A_{\eta}}$  is canonically identified with a linear system. The map  $\eta$  induces a natural inclusion  $\iota_{\eta} : \mathbb{C}^A \rightarrow \mathbb{C}^{A_{\eta}}$  given by

$$(22) \quad \iota_{\eta} \left( \sum_{\alpha \in A} c_{\alpha} e_{\alpha} \right) = \sum_{\alpha \in A} c_{\alpha} e_{(\alpha, \eta(\alpha))}.$$

**Definition 2.7.** A degenerating family of  $(\mathcal{X}_Q, \mathcal{Y}_S)$  is a toric hypersurface  $(\mathcal{X}, \mathcal{Y})$  equivalent to a pair  $(\mathcal{X}_{\eta}, \mathcal{Y}_{\iota_{\eta}(s')})$  for some a defining function  $\eta$  of a regular subdivision  $S$  of  $(Q, A)$  and a very full section  $s'$ .

We note that the stack  $\mathcal{X}_{\eta}$  admits a morphism  $F_{\eta} : \mathcal{X}_{\eta} \rightarrow \mathbb{C}$ . Taking  $\mathbb{C}$  to be the stacky fan given by  $(\mathbb{Z}, \mathbb{Z}, \mathbb{N}, Id)$ , we may describe  $F_{\eta}$  as a map  $(f_1, f_2)$  of stacky fans

$$(23) \quad \begin{array}{ccc} \mathbb{Z}^{\overline{Q}_{\eta}} & \xrightarrow{\beta_{\overline{Q}_{\eta}}} & (\Lambda \oplus \mathbb{Z})^{\vee} \\ f_1 \downarrow & & f_2 \downarrow \\ \mathbb{Z} & \xrightarrow{Id} & \mathbb{Z}. \end{array}$$

Here, for every  $b \in \overline{Q}_{\eta}^h$ ,  $f_1(e_b) = 0$  while for  $b_i \in \overline{Q}_{\eta}^v$  corresponding to  $(Q_i, A_i)$ ,  $f_1(e_{b_i}) = c_{\eta,i}$ . The homomorphism  $f_2$  is simply projection to the  $\mathbb{Z}$  factor. It is not hard to see then that the fiber of  $(\mathcal{X}_{\eta}, \mathcal{Y}_{\iota_{\eta}(s)})$  over  $1 \in \mathbb{C}^*$  is equivalent to

$(\mathcal{X}_Q, \mathcal{Y}_s)$ . On the other hand, the fiber over zero can be seen to have components which are equivalent to the toric varieties  $(\mathcal{X}_{A_i}, \mathcal{Y}_{s|_{A_i}})$ . This justifies the definition given above.

It is useful to view the morphism  $F_\eta$  from the moment map perspective as well. Here we have that  $\mu_{L_{\overline{Q}_\eta}}^{-1}(\omega) \subset \mathbb{C}^{\overline{Q}_\eta}$  defines the stack  $\mathcal{X}_\eta$  after taking the quotient by  $\mathbb{T}_{L_{\overline{Q}_\eta}}$ . Observe that the map  $F_\eta$  then can be defined on  $\mathbb{C}^{\overline{Q}_\eta}$  as the map

$$(24) \quad \tilde{F}_\eta(z_1, \dots, z_{|\overline{Q}_\eta|}) = \prod_{i \in \overline{Q}_\eta^v} z_i^{c_{\eta,i}}.$$

In other words,  $\tilde{F}_\eta$  is invariant with respect to the  $\mathbb{T}_{L_{\overline{Q}_\eta}}$  action and descends to  $F_\eta$  on the quotient  $\mathcal{X}_\eta = [\mu_{L_{\overline{Q}_\eta}}^{-1}(\omega)/\mathbb{T}_{L_{\overline{Q}_\eta}}]$ .

In general, the marking  $A$  should be thought of as a set specifying the non-zero coefficients of a given section. Let  $A_v \subset A$  be the set of vertices of  $Q$  and call any toric hypersurface  $(\mathcal{X}_Q, \mathcal{Y}_s)$  if  $s \in (\mathbb{C}^*)^{A_v} \times \mathbb{C}^{A-A_v}$  and very full if  $s \in (\mathbb{C}^*)^A$ .

**Definition 2.8.** A toric degeneration of a toric hypersurface  $(\mathcal{X}_Q, \mathcal{Y}_s)$  is a fiber  $(F_\eta^{-1}(0), F_\eta^{-1}(0) \cap \mathcal{Y})$  where  $F_\eta : \mathcal{X} \rightarrow \mathbb{C}$  is the projection associated to a degenerating family of  $(\mathcal{X}_Q, \mathcal{Y}_s)$ . If  $t \in \mathbb{C}$ , we write  $\mathcal{Z}_\eta(t)$  for the fiber  $F_\eta^{-1}(t)$ .

**2.3. Secondary and Lafforgue stacks.** In this section we give an explicit formulation of several auxiliary stacks related to  $A$ . One stack we obtain is close to those defined in [45] and [4], but with a universal line bundle and section.

We start by setting up more notation and recalling several general results from [46]. Again we assume  $A \subset \Lambda$  is a finite set which affinely spans  $\Lambda \otimes \mathbb{R}$  and let  $A^e = \text{Lin}_{\mathbb{N}}\{(\alpha, 1) \in \Lambda \oplus \mathbb{Z} : \alpha \in A\}$  and  $A^e_k = \{(v, k) \in A^e\}$ . We note that  $\overline{A^e} = \{(b, -n_b) : b \in \overline{A}\}$  and  $\mathcal{X}_{A^e}$  is the cone of  $\mathcal{X}_A$ . Recall that the fundamental sequence associated to  $A^{e_1}$  is

$$(25) \quad 0 \rightarrow L_{A^{e_1}} \xrightarrow{\alpha_{A^{e_1}}} \mathbb{Z}^{A^{e_1}} \xrightarrow{\beta_{A^{e_1}}} \Lambda \oplus \mathbb{Z} \rightarrow K_{A^{e_1}} \rightarrow 0.$$

A marked polytope  $(Q, A)$  will be referred to as a simplex if  $Q$  is a simplex and  $A$  is its set of vertices. By a regular triangulation of  $A$  we mean a regular subdivision  $S = \{(Q_i, A_i) : i \in I\}$  such that every  $(Q_i, A_i)$  is a simplex. The secondary polytope  $\Sigma(A)$  of  $A$  is then defined to be the convex hull

$$(26) \quad \Sigma(A) = \text{Conv}\{\varphi_T : T \text{ a regular triangulation of } A\} \subset \mathbb{Z}^A$$

where, if  $T = \{(Q_i, A_i) : i \in I\}$  is such a triangulation,

$$(27) \quad \varphi_T = \sum_{\alpha \in \cup A_i} \left( \sum_{\alpha \in A_i} \text{Vol}(Q_i) \right) e_\alpha \in \mathbb{Z}^A.$$

The next cited theorem connects the secondary polytope to the linear system  $\mathcal{L}_A$ .

**Theorem 2.9** ([46], 10.1).

- (i) *The Newton polytope of  $E_A$  is  $\Sigma(A)$ .*
- (ii)  $E_A(f) = \prod_{Q' < Q} \delta_{A \cap Q'}(f)^{i(\Gamma, A) \cdot u(S/\Gamma)}.$

The product is over all marked faces  $Q'$  of  $Q$  and the exponent  $i(Q', A) \cdot u(S/Q')$  equals the multiplicity of any point on the orbit associated to  $Q'$  in the  $A$ -philosophy formulation of toric varieties. The factors are given as follows. Take the semigroup

$S$  to be that generated by  $A^e$  and for any face  $Q' \subset Q$ , take  $\text{Lin}_{\mathbb{R}}(Q')$  and  $\text{Lin}_{\mathbb{Z}}(Q')$  to be the  $\mathbb{R}$  linear and  $\mathbb{Z}$  linear span of  $\tilde{Q}' = \{(\alpha, 1) : \alpha \in Q' \cap A\}$  respectively. Then the index  $i(Q', A)$  is set to equal  $[\Lambda \oplus \mathbb{Z} \cap \text{Lin}_{\mathbb{R}}(Q') : \text{Lin}_{\mathbb{Z}}(A \cap Q')]$ . The term  $u(S/Q')$  denotes the subdiagram volume of the semigroup  $S/Q'$  which is the image of  $S$  in  $\Lambda \oplus \mathbb{Z}/(\Lambda \oplus \mathbb{Z} \cap \text{Lin}_{\mathbb{R}}(Q'))$ . We prefer this formulation over simply writing the multiplicity since our definition of a toric stack associated to a polytope does not coincide with the one given in [46]. However, we always have a dominate map from our  $\mathcal{X}_Q$  to theirs, namely, the map associated to the linear system given by  $A$ .

Perhaps more familiar, from section 2.2, than the secondary polytope is the secondary fan. The cones of this fan are the cones  $C_{\mathbb{R}}(S)$  for all regular subdivisions  $S$ . We write  $\mathcal{F}_{\Sigma(A)} \subset (\mathbb{R}^A)^\vee$  for the fan and cite the following theorem.

**Theorem 2.10** ([46], 7.1).

- (i) *The secondary polytope  $\Sigma(A)$  has a single point as its image under  $\beta_{A^e}$ .*
- (ii) *The fan  $\mathcal{F}_{\Sigma(A)}$  is the normal fan of  $\Sigma(A)$ .*

In more detail, we have that  $\beta_{A^e}(\Sigma(A)) = (\delta_Q, (d+1)\text{Vol}(Q))$  where  $\delta_Q = (d+1) \int_Q x \, dx$  is the dilated centroid of  $Q$ . We will define several stacks associated to  $\Sigma(A)$  utilizing techniques from 2.1. Since  $\Sigma(A)$  does not affinely span  $\mathbb{R}^A$ , we cannot define  $\mathcal{X}_{\Sigma(A)}$  as before. Instead, choose any  $v \in \mathbb{Z}^A$  for which  $\beta_{A^e}(v) = \delta_Q$  and let

$$(28) \quad \Sigma_v(A) = \{w \in L_A : \alpha_A(w) + v \in \Sigma(A)\} \subset L_A.$$

The stack  $\mathcal{X}_{\Sigma_v(A)}$  is clearly independent of the choice of  $v$ , and we denote it simply  $\mathcal{X}_{\Sigma(A)}^r$ . Let us detail the stacky fan in this case. Recall that  $\overline{\Sigma_v(A)} \subset L_A^\vee$  denotes the supporting hyperplane primitives for  $\Sigma_v(A)$ . By [46, 7.2], the set of supporting hyperplanes is  $\{b_S : S \text{ a coarse subdivision}\}$ . By definition, a coarse subdivision is a regular subdivision that is not a refinement of any non-trivial regular subdivision. A collection  $J \subset \overline{\Sigma_v(A)}$  is in the abstract simplicial complex  $\mathcal{B}$  associated to  $\Sigma_v(A)$  if and only if there is a regular subdivision  $S$  refining the coarse subdivisions  $\{S_b : b \in J\}$ . These structures give the stacky fan  $\Sigma_{\Sigma_v(A)} = (\overline{\mathbb{Z}^{\Sigma_v(A)}}, L_A^\vee, \Sigma_{\mathcal{B}}, \beta_{\overline{\Sigma_v(A)}})$  for  $\mathcal{X}_{\Sigma(A)}^r$ .

To obtain more control over the hypersurfaces in  $\mathcal{X}_Q$  and their degenerations, we will need a more nuanced secondary stack than  $\mathcal{X}_{\Sigma(A)}^r$ . Instead of working around the fact that  $\Sigma(A)$  does not span  $\mathbb{R}^A$ , we extend the polytope  $\Sigma(A)$  to a polyhedron  $\Theta_p(A)$  and apply constructions from section 2.1. This yields a stack  $\mathcal{X}_{\Theta(A)}$  which we call the Lafforgue stack of  $A$  as the toric variety associated to it equals the Lafforgue variety as defined in [45].

**Definition 2.11.** Let  $\Delta_t^A = \{\sum_{\alpha \in A} c_\alpha e_\alpha : c_\alpha \geq 0, \sum c_\alpha = t\}$  be a simplex in  $\mathbb{R}^A$  and  $\Delta_{\geq t}^A = \cup_{s \geq t} \Delta_s^A$ .

- (i) The Lafforgue polytope  $\Theta(A)$  of  $A$  is the Minkowski sum  $\Sigma(A) + \Delta_1^A$ ,
- (ii) The Lafforgue polyhedron  $\Theta_p(A)$  of  $A$  is defined as the Minkowski sum  $\Sigma(A) + \Delta_{\geq 1}^A$ .

To justify the name of these polyhedra, we recall the construction by Lafforgue ([45], [31]) of a fan  $\mathcal{F}_{\Theta_p(A)}$  which refines the secondary fan  $\mathcal{F}_{\Sigma(A)}$ . Given a regular subdivision  $S = \{(Q_i, A_i) : i \in I\}$  and a marked face  $(Q_J, A_J)$  of the subdividing

polytopes, where  $A_J := \cap_{j \in J} A_i$  for some  $J \subset I$ , we define the cone

$$(29) \quad C_{\mathbb{R}}(S, A_J) = \{\eta \in C_{\mathbb{R}}(S) : \eta(e_\alpha) \leq \eta(e_\beta) \text{ for all } \alpha \in A_J, \beta \in A\}.$$

We call the pair  $(S, A_J)$  a pointed subdivision and when  $A_J = \{\alpha\}$ , we simply write  $C_{\mathbb{R}}(S, \alpha)$ . It is clear that  $C_{\mathbb{R}}(S, A_J) \subset C_{\mathbb{R}}(S', A'_J)$  if and only if  $S'$  refines  $S$  and  $A_J \supset A'_J$ . In this case we write  $(S', A'_J) \preceq (S, A_J)$ . By definition, the fan  $\mathcal{F}_{\Theta(A)}$  consists of the cones  $C_{\mathbb{R}}(S, J)$ . For certain classes of sets  $A$ , Lafforgue has shown that the toric variety associated to this fan yields a parameter space for toric degenerations of the variety  $X_A$ . However, this paper is concerned primarily with hypersurface degenerations, so in order to relate this work to ours, we require a line bundle on the associated variety. Furthermore, to preserve information on toric isomorphisms, we wish to consider the toric stack construction along the lines of section 2.1. We set out to accomplish both of these aims through the use of  $\Theta_p(A)$  and  $\Theta(A)$ . First we verify:

**Proposition 2.12.** *The fan  $\mathcal{F}_{\Theta(A)}$  is the normal fan of the polytope  $\Theta(A)$ .*

*Proof.* It is not hard to see that  $\mathcal{F}_{\Theta(A)}$  is determined by the refinement on the maximal cones of  $\mathcal{F}_{\Sigma(A)}$ , i.e. the cones  $C_{\mathbb{R}}(S, A_J)$  can be described as intersections of  $C_{\mathbb{R}}(T_j, \alpha_k)$  where  $T_j$  is a regular triangulation refining  $S$  and  $A_J = \{\alpha_k : k \in K\}$ . It thus suffices to show that  $C_{\mathbb{R}}(T, \alpha)$  is the normal cone of the vertex

$$(30) \quad \varphi_{(T, \alpha)} := \varphi_T + e_\alpha.$$

Suppose  $\psi \in C_{\mathbb{R}}(T, \alpha)$ ,  $T'$  is any triangulation and  $(T')^0$  is the union of the markings of the simplices in  $T'$ . Then for any  $\beta_1, \beta_2 \in (T')^0$  we have

$$(\psi, \varphi_{(T', \beta_1)}) - (\psi, \varphi_{(T', \beta_2)}) = (\psi(\beta_1) - \psi(\beta_2)).$$

Thus  $(\psi, \phi_{(T', \beta_1)}) \geq (\psi, \phi_{(T', \beta_2)})$  iff  $\psi(\beta_1) \geq \psi(\beta_2)$ . In other words, the minimum that  $\psi$  obtains amongst the vertices of  $\Theta(A)$  coming from  $T' + \Delta_1^A$  is the pointed triangulation  $(T', \beta)$  where  $\psi(\beta)$  is minimal. Choose  $\beta \in (T')^0$  to be such an element. By the definition of  $C_{\mathbb{R}}(T, \alpha)$ , we have  $\psi(\alpha) \leq \psi(\gamma)$  for any  $\gamma \in A$ . Using the result that the secondary fan is dual to the secondary polytope, we have

$$\begin{aligned} (\psi, \phi_{(T, \alpha)}) &= (\psi, \phi_T) + \psi(\alpha), \\ &\leq (\psi, \phi_T) + \psi(\beta), \\ &\leq (\psi, \phi_{T'}) + \psi(\beta), \\ &= (\psi, \phi_{(T', \beta)}). \end{aligned}$$

Thus  $C_{\mathbb{R}}(T, \alpha) \subseteq N_{\phi_{(T, \alpha)}}(\Theta(A))$ . For the converse, one simply observes that both the dual fan to  $\Theta(A)$  and the fan  $\mathcal{F}_{\Theta(A)}$  are complete fans of  $\mathbb{R}^A$  with  $C(T, \alpha)$  and  $N^{\phi_{(T, \alpha)}}$  both  $|A|$ -dimensional cones. As the number of vertices of  $\Theta(A)$  equals the number of maximal cones in  $\mathcal{F}_{\Theta_p(A)}$ , one inclusion then implies the other.  $\square$

This proposition gives us a polarization for the variety associated to the Lafforgue fan. However, if we wanted to obtain a polytope spanning  $\mathbb{R}^A$ , we have missed the mark by one dimension. As in the case of the secondary polytope, we could restrict to the subspace spanned by  $\Sigma(A)$ . However, it is more natural to consider the polyhedron  $\Theta_p(A) \subset \mathbb{R}^A$  and a scaled variant of its associated polarized stack  $\mathcal{X}_{\Theta_p(A)}$  as in section 2.1. In other words, we will define a stacky fan

$$(31) \quad \tilde{\Sigma}_{\Sigma(A)} = \left( \mathbb{Z}^{\overline{\Theta_p(A)}}, (\mathbb{Z}^A)^\vee, \beta, \Sigma_{\Theta_p(A)} \right)$$

where the lattices and fan are equal to those for the stacky fan of  $\Theta_p(A)$  as in definition 2.5, but  $\beta(e_b) = c_b e_b$  for some constants  $c_b \in \mathbb{N}$  which we now define.

We start by examining the supporting hyperplane sections  $\overline{\Theta_p(A)}$  and the associated fan  $\Sigma_{\overline{\Theta_p(A)}}$  in  $\mathbb{R}^{\overline{\Theta_p(A)}}$ . From the homogeneity of  $\Sigma(A)$  and the definition of  $\Theta_p(A)$ , it is easy to see that  $\varrho_A := \sum_{\alpha \in A} e_\alpha^\vee \in \overline{\Theta_p(A)}$ . We take  $B = \overline{\Theta_p(A)} - \{\varrho_A\}$  and observe that all  $b \in B$  define supporting hyperplanes of  $\Theta(A)$ . By proposition 2.12, these elements may be identified with the minimal, two dimensional cones of  $\mathcal{F}_{\Theta_p(A)}$  and, indeed, any element of  $B$  is contained in a corresponding normal cone. To identify the elements of  $B \subset (\mathbb{Z}^A)^\vee$  more precisely, observe that the minimal cones of  $\mathcal{F}_{\Theta_p(A)}$  are described by pointed subdivisions  $(S, A_J)$  which are maximal with respect to the partial order  $\preceq$ . Note that such subdivisions  $(S, A_J)$  may be partitioned into two types, those for which  $S = \{(Q_i, A_i) : i \in I\}$  is a coarse subdivision with  $A_J = A_i$  for some  $i \in I$ , and those with  $S = \{(Q, A)\}$  and  $A_J = Q' \cap A$  with  $Q'$  a facet of  $Q$ . We write  $B^v$  for the elements corresponding to the former case and  $B^h$  for those for the latter. The following proposition follows from this discussion and results in [46, 7.1].

**Proposition 2.13.** *The hyperplane primitives  $\overline{\Theta_p(A)}$  are the disjoint union  $\{\varrho_A\} \cup \overline{\Theta_p(A)}^h \cup \overline{\Theta_p(A)}^v$ .*

- (i) *If  $b \in \overline{\Theta_p(A)}^h$  corresponds to  $(\{(Q, A_J)\}, A_J)$  and  $\tilde{b} \in \overline{A^e}$  is the defining hyperplane function for  $A_J \oplus \{1\}$  then there is a  $c_b \in \mathbb{N}$  such that  $b = c_b^{-1} \beta_{A_{e_1}}^\vee(\tilde{b})$ .*
- (ii) *If  $b \in \overline{\Theta_p(A)}^v$  corresponds to  $(S, A_J)$ , then  $b = \eta_{(S, A_i)} \in C_{\mathbb{Z}}^\circ(S)$  is the primitive defining function for  $S$  satisfying  $\eta_{(S, A_i)}|_{A_i} = 0$ .*

Using this proposition and letting  $c_b = 1$  for all  $b \notin \overline{\Theta_p(A)}^h$ , we take  $\beta(e_b) = c_b e_b$  and write the following definition.

**Definition 2.14.**

- (i) The total Lafforgue stack of  $A$  is  $\mathcal{X}_{\Theta_p(A)} := \mathcal{X}_{\Sigma_{\Theta_p(A)}}$ .
- (ii) The Lafforgue stack of  $A$  is  $\mathcal{X}_{\Theta(A)} := D_{\varrho_A}$ .
- (iii) The universal line bundle  $\mathbf{O}_A(1)$  on  $\mathcal{X}_{\Theta(A)}$  is the dual  $\mathcal{X}_{\Theta_p(A)}^\vee$ .
- (iv) The universal section  $s_A \in H^0(\mathcal{X}_{\Theta(A)}, \mathcal{O}_A(1))$  is the sum  $\sum_{\alpha \in A} x_\alpha$ .
- (v) The universal hyperplane  $\mathcal{V}_A \subset \mathcal{X}_{\Theta(A)}$  is defined to be the zero locus of  $s_A$ .

Observe that we may take the star of  $\varrho_A$  in  $\Sigma_{\Theta_p(A)}$  which yields a fan  $\Sigma_{\Theta(A)}$  in  $\mathbb{R}^B$  combinatorially equivalent to the Lafforgue fan. The map  $\beta_{\Theta(A)} : \mathbb{Z}^B \rightarrow (\mathbb{Z}^A)^\vee / \mathbb{Z} \cdot \varrho$  then defines the stacky fan  $\Sigma_{\Theta(A)} := (\mathbb{Z}^B, (\mathbb{Z}^A)^\vee / (\varrho_A), \Sigma_{\Theta(A)}, \beta_{\Theta(A)})$  which gives an alternative description of  $\mathcal{X}_{\Theta(A)}$ . The advantage of detailing the total Lafforgue stack is twofold. First,  $\mathcal{X}_{\Theta_p(A)}$  is the total space of a line bundle  $\mathcal{L}_{\Theta_p(A)}$  over  $\mathcal{X}_{\Theta(A)}$  and second the polyhedron description of  $\mathcal{X}_{\Theta_p(A)}$  yields an additional line bundle  $\mathcal{O}_{\Theta_p(A)}(1)$ .

To give the last definition of this section we examine the diagram

$$(32) \quad \begin{array}{ccc} \mathbb{Z}^{\overline{\Theta_p(A)}} & \xrightarrow{\beta_{\overline{\Theta_p(A)}}} & (\mathbb{Z}^A) \\ p_1 \downarrow & & \alpha_A^\vee \downarrow \\ \mathbb{Z}^{\overline{\Sigma_v(A)}} & \xrightarrow{\beta_{\overline{\Sigma_v(A)}}} & L_A^\vee. \end{array}$$

Here  $p_1$  is defined as sending all basis elements from  $\{\varrho_A\} \cup \overline{\Theta_p(A)}^h$  to zero and  $b \in \overline{\Theta_p(A)}^v$  corresponding to  $(S, A)$  to  $m_b e_{b_S}$  where  $m_b b_S = \alpha_A^\vee(b)$ . This induces a flat morphism  $p = (p_1, \alpha_A^\vee)$  from  $\mathcal{X}_{\Theta(A)}$  to  $\mathcal{X}_{\Sigma(A)}^r$ .

**Definition 2.15.** The secondary stack is defined to be  $\mathcal{X}_{\Sigma(A)} := \mathcal{X}_p^{\rightarrow}$  and the map  $p^{\rightarrow}$  will be written as  $\pi : \mathcal{X}_{\Theta_p(A)} \rightarrow \mathcal{X}_{\Sigma(A)}$ . The fine quotient of  $p$  will be called the full secondary stack  $\mathcal{X}_{\Sigma(A)}^f := \mathcal{X}_p^f$ . Write  $\mathcal{E}_A \subset \mathcal{X}_{\Sigma(A)}$  for the pullback of the zero locus of  $E_A$ . Using the coefficients of the  $A$ -determinant  $E_A$  for the section  $E_A^s \in \mathcal{O}_{\Sigma(A)}(1)$ .

We note that the full secondary stack has strata with infinite stabilizer groups, while all such groups in the secondary stack are finite.

**2.4. Toric hypersurface moduli.** In this section we show that there is an affine open set  $\mathcal{V}_A \subset \mathcal{X}_{\Sigma(A)}$  which is identified with the moduli space of full hypersurfaces in  $\mathcal{V}_A$ . We emphasize here that by moduli space, we mean toric hypersurfaces up to toric equivalence, not up to isomorphism. This toric moduli space  $\mathcal{V}_A$  is an affine DM stack and is therefore much easier to control. We then show that the pullback  $\mathcal{V}_A \subset \mathcal{X}_{\Theta(A)}$  along the inclusion yields a universal toric hypersurface over  $\mathcal{V}_A$ . Finally, we prove that any toric degeneration  $F_\eta : \mathcal{X}_\eta \rightarrow \mathbb{C}$  can be obtained by pulling back  $\mathcal{X}_{\Theta(A)}$  along a map  $\rho_\eta : \mathbb{C} \rightarrow \mathcal{X}_{\Sigma(A)}$  where 0 is sent to the compactifying divisor  $\mathcal{X}_{\Sigma(A)} - \mathcal{V}_A$ . This gives meaning to the notion of  $\mathcal{X}_{\Sigma(A)}$  as a moduli for hypersurface degenerations. We note that many of the proofs involved in this subsection may be drastically simplified if we add assumptions such as  $\mathcal{X}_Q$  being a smooth toric variety or even  $\mathcal{X}_Q$  having trivial generic isotropy. However, in our context, it is necessary to apply the propositions to more general toric stacks, so we add as few conditions as possible.

Upon examining the constructions of the previous sections, one observes that the fundamental sequences of  $\overline{Q}$ ,  $\overline{\Sigma(A)}$ ,  $\overline{\Theta_p(A)}$  all come into play, but those of the original sets are largely absent. We remedy this situation for  $A^e$  here. First recall that  $\mathcal{O}_A(1) = \mathcal{O}(D_{\gamma_A})$  where  $\gamma_A = \sum_{b \in \overline{Q}} n_b e_b^\vee \in \mathbb{Z}^{\overline{Q}}$  and the maximal torus acting on  $\mathcal{O}_A(1)$  is naturally identified with  $(\mathbb{Z}^d)^\vee \otimes \mathbb{C}^*$ . The choice of equivariant divisor means the action of  $(\mathbb{Z}^d)^\vee \otimes \mathbb{C}^*$  on  $\mathcal{X}_Q$  lifts to one on  $\mathcal{L}_A \subset H^0(\mathcal{X}_Q, \mathcal{O}_A(1))$ . Including scaling into the action, we obtain an  $(\mathbb{Z}^d \oplus \mathbb{Z})^\vee \otimes \mathbb{C}^*$  action. The characters of this action are easily seen to correspond to the elements of  $A^e \subset \mathbb{Z}^d \oplus \mathbb{Z}$ . This leads to the definition:

**Definition 2.16.** The  $A$ -linear system quotient stack is  $\mathcal{X}_{\mathcal{L}_A}$  with stacky fan:

$$(33) \quad \Sigma_{\mathcal{L}_A} = \left( (\mathbb{Z}^{A^e})^\vee, \Lambda_{A^e}, \Sigma, \alpha_{A^e}^* \right)$$

where  $\Sigma$  is the fan with unique maximal cone  $(\mathbb{Z}_{\geq 0}^{A^e})^\vee$ .

While it is an elementary check to see that this gives the stack  $[\mathcal{L}_A / (\mathbb{Z}^d \oplus \mathbb{Z})^\vee \otimes \mathbb{C}^*]$  corresponding to hypersurfaces up to toric equivalence, it is slightly less obvious that there is a smooth affine Deligne-Mumford (or DM) substack of full subsections.

**Proposition 2.17.** The substack  $\mathcal{V}_A \subset \mathcal{X}_{\mathcal{L}_A}$  defined from the stacky fan

$$(34) \quad \left( (\mathbb{Z}^{A^e})^\vee, \Lambda_{A^e}, \Sigma', \alpha_{A^e}^* \right)$$



with  $\Sigma'$  the fan with unique maximal cone given by  $(\mathbb{Z}^{A_v^e})^\vee$  is an affine DM substack.

*Proof.* Write  $\mathbb{Z}^{A^e} = \mathbb{Z}^{A_v^e} \oplus \mathbb{Z}^{A_{nv}^e}$  where  $A_{nv}$  and define the map

$$(35) \quad g_1 : \mathbb{Z}^{A^e} \rightarrow L_{A_v^e}^\vee \oplus \mathbb{Z}^{A_{nv}^e}$$

as  $g_1 = (\alpha_{A_v^e}^\vee, 1)$ . Note that  $g_1$  maps  $\Sigma'$  to an image fan  $\Sigma''$ .

Let  $g_2$  be the identity on  $L_{A^e}^\vee \oplus \text{Ext}^1(K_{A^e}, \mathbb{Z})$  and

$$(36) \quad \Sigma' = (L_{A_v^e}^\vee \oplus \mathbb{Z}^{A_{nv}^e}, L_{A^e}^\vee \oplus \text{Ext}^1(K_{A^e}, \mathbb{Z}), \Sigma'', \beta)$$

where  $\beta$  equals  $\alpha^*$  on  $\mathbb{Z}^{A_{nv}^e}$  and is the natural inclusion on  $L_{A_v^e}^\vee$  induced by the covariance of the fundamental sequence with respect to inclusion. Take  $\mathcal{X}$  to be the associated stack and note that since  $g_2$  is an isomorphism and  $\beta \circ g_1$  is surjective  $g : \mathcal{V}_A \rightarrow \mathcal{X}$  is an isomorphism.

Since the kernel of  $\beta$  is trivial and  $\Sigma''$  gives a product of affine spaces and complex tori, the stack  $\mathcal{X}$ , and therefore  $\mathcal{V}_A$ , is an affine DM stack.  $\square$

We write  $\tilde{\mathcal{V}}_A$  for the open toric variety given by  $\Sigma''$  and  $\psi_A : \tilde{\mathcal{V}}_A \rightarrow \mathcal{V}_A$  for the étale map with Galois group  $G_A = \text{Hom}(K_{A_v^e}, \mathbb{C}^*)$ . This map should be thought of as a quotient by torus automorphisms. Note that  $G_A$  also acts naturally on  $\mathcal{X}_Q$  by taking  $\text{Hom}(-, \mathbb{C}^*)$  of the fundamental exact sequence for  $A_v^e$  and letting the additional  $\mathbb{C}^*$  act trivially. We take  $\tilde{\mathcal{U}}_A = \tilde{\mathcal{V}}_A \times \mathcal{X}_Q$  and define the universal toric stack over  $\mathcal{V}_A$  to be the quotient

$$(37) \quad \mathcal{U}_A := [\tilde{\mathcal{U}}_A / G_A].$$

Let  $\tilde{s} \in H^0(\tilde{\mathcal{U}}_A, \mathcal{O}_{\tilde{\mathcal{V}}} \otimes \mathcal{O}_A(1))$  be the tautological section. It is clearly invariant with respect to the  $G_A$  action and we consider the pair  $\mathcal{W}_A \subset \mathcal{U}_A$  to be the universal toric hypersurface  $\tilde{s} = 0$  over  $\mathcal{V}_A$ .

The following theorem shows that the secondary stack from the previous section is a compactification of the moduli stack  $\mathcal{V}_A$  of full sections and that the Lafforgue stack along with the section  $s_A$  pulls back to the universal toric hyperplane. The compactifying strata correspond to reasonable degenerations of  $\mathcal{X}_Q$ . This is in analogy to the moduli space of curves and their stable compactifications which served as motivation for the definition.

**Theorem 2.18.** *There is an open embedding  $i : \mathcal{V}_A \rightarrow \mathcal{X}_{\Sigma(A)}$ . If  $p \notin i(\mathcal{V}_A)$ , then  $p$  is in a boundary divisor  $D_S$  where  $S = \{(Q_i, A_i) : i \in I\}$  is a coarse subdivision and  $|I| > 1$ . The pullback of  $\mathcal{V}_A \subset \mathcal{X}_{\Theta(A)}$  is equivalent to the universal toric hypersurface over  $\mathcal{W}_A \subset \mathcal{U}_A$ .*

*Proof.* We first examine the open substack  $\mathcal{X}_{\Theta(A)}^\circ$  defined by the subfan consisting of only the coarse subdivisions  $\{e_\alpha^\vee\}_{\alpha \in A_{nv}}$  and the elements in  $\overline{\Theta_p(A)}^h$ . More precisely, take

$$(38) \quad \Sigma_u = (\mathbb{Z}^{\overline{\Theta_p(A)}^h} \oplus \mathbb{Z}^{A_{nv}} \oplus \mathbb{Z}^{A_v}, (\mathbb{Z}^A)^\vee, \Sigma_u, \beta_{\overline{\Theta_p(A)}^h}),$$

where  $\Sigma_u$  is supported on the span  $\mathbb{Z}^{\overline{\Theta_p(A)}^h} \oplus \mathbb{Z}^{A_{nv}}$  where it is the restriction of the normal to  $\Theta_p(A)$ . We note the obvious fact that  $\beta_{\overline{\Theta_p(A)}^h}$  is onto. The image  $\mathcal{X}_{\Sigma(A)}^\circ = \pi(\mathcal{X}_{\Theta(A)}^\circ)$  is easily seen to be equal to the colimit stack of  $p : \mathcal{X}_{\Theta(A)}^\circ \rightarrow \mathcal{X}_{\Sigma(A)}^r$ . We

make the following claim from which the first two assertions in the theorem easily follow.

*Claim:* The stack  $\mathcal{X}_{\Sigma(A)}^\circ$  is an open substack isomorphic to  $\mathcal{V}_A$ .

We show this claim by observing that the image of every horizontal dual in  $\overline{\Theta_p(A)}^h$  is zero under  $p$  and that the image of  $e_\alpha^\vee$  is simply  $\alpha_A^\vee(e_\alpha^\vee) \in L_A^\vee$ . As each supporting hyperplane for the coarse subdivisions given by non-vertex points uniquely map to their image in the secondary fan. This, along with the fact that  $p$  is flat, implies that  $p_1|_{\mathbb{Z}^A}$  is a bijection onto its image which we write as  $\Gamma \approx (\mathbb{Z}^A)^\vee$  with fan the image of  $\Sigma_u$  which has the maximal cone given by basis elements corresponding to non-vertex. We have the pushout diagram

$$(39) \quad \begin{array}{ccc} \mathbb{Z}^{\overline{\Theta_p(A)}^h} \oplus \mathbb{Z}^{A_v} \oplus \mathbb{Z}^{A_{nv}} & \xrightarrow{\beta_{\overline{\Theta_p(A)}}} & (\mathbb{Z}^A)^\vee \\ p \downarrow & & \downarrow \kappa \\ \Gamma & \xrightarrow{\alpha_A^*} & \Lambda_{A^\vee}. \end{array}$$

To see this, observe that since  $p$  is an isomorphism on  $\mathbb{Z}^A$  and sends  $\mathbb{Z}^{\overline{\Theta_p(A)}^h}$  to zero, the map  $\kappa$  is simply the cokernel of  $\beta_{\overline{\Theta_p(A)}}$ . But by proposition 2.13, we have that the image of  $\beta_{\overline{\Theta_p(A)}}$  restricted to  $\mathbb{Z}^{\overline{\Theta_p(A)}}$  is the image of  $\beta_A^\vee$  which has the indicated cokernel from the dual fundamental sequence for  $A$ . The described stacky fan data obtained on the bottom of the diagram defines the colimit stack and is identical to the stacky fan defining  $\mathcal{V}_A$ . This proves the claim.

To see the second part of the theorem, one simply identifies  $\mathcal{W}_A \subset \mathcal{U}_A$  with  $\mathcal{Y}_A \cap \mathcal{X}_{\Theta(A)}^\circ \subset \mathcal{X}_{\Theta(A)}^\circ$  through a similar argument.  $\square$

Finally, we describe the points on the compactifying divisor.

**Theorem 2.19.** *Every degenerating family  $(\mathcal{X}, \mathcal{Y})$  of a hypersurface  $(\mathcal{X}_Q, \mathcal{Y}_s)$  is represented by a map  $v : \mathbb{C} \rightarrow \mathcal{X}_{\Sigma(A)}$ .*

*Proof.* Let  $\eta \in \mathbb{Z}^A$  be a primitive defining function for the family  $(\mathcal{X}, \mathcal{Y})$  corresponding to the subdivision  $S = \{(Q_i, A_i) : i \in I\}$  with  $|I| > 1$ . Define a map  $v_\eta : \mathbb{N} \rightarrow L_A^\vee$  by taking the  $\alpha^*(\eta)$ . The stacky fan  $\Sigma = (\Lambda_1, \Lambda_2, \Sigma, \beta)$  occurring in the fiber product  $\mathbb{C} \times_{\tilde{v}_\eta} \times_\pi \mathcal{X}_{\Theta(A)}$  has  $\Lambda_1 \approx (\mathbb{Z}^d)^\vee \oplus \mathbb{Z}$  from the Cartesian diagram

$$(40) \quad \begin{array}{ccc} (\mathbb{Z}^d)^\vee \oplus \mathbb{Z} \cdot \eta & \xrightarrow{\psi} & (\mathbb{Z}^A)^\vee / (\sum_{\alpha \in A} e_\alpha^\vee) \\ p_2 \downarrow & & \downarrow \alpha^* \\ \mathbb{Z} & \xrightarrow{v_\eta} & \Lambda_{A^\vee}, \end{array}$$

where  $\psi := \tilde{\beta}_A^\vee \oplus inc$  and  $\tilde{\beta}_A = proj \circ \beta_A|_{(\mathbb{Z}^d)^\vee}$ . To find  $\Lambda_1$  and  $\Sigma$ , we let  $\Sigma_\eta$  be the fan obtained by intersecting  $\beta_A^\vee \oplus inc((\mathbb{R}^d)^\vee \oplus \mathbb{R}_{\geq 0})$  with the Lafforgue fan and  $T_\eta$  the primitive generators of its one cones. The map  $\beta_\eta : \mathbb{Z}^{T_\eta} \rightarrow \Lambda_2$  by evaluation of primitives gives the stack  $\Sigma_\eta = (\mathbb{Z}^{T_\eta}, \Lambda_2, \Sigma_\eta, \beta_\eta)$ . For every  $\tau \in \overline{B}_\eta$ , we have that

$$\tau = \sum_{b \in \sigma(1) \cap \overline{\Theta_p(A)}} c_b b$$

for some maximal cone  $\sigma$ . Let  $g_1(e_\tau) = \sum_{b \in \sigma(1)} c_b e_b$ . It is not hard to see that the map  $g = (g_1, 1)$  then induces an equivalence between  $\Sigma_\eta$  and the pullback  $\Sigma$ .

To see that  $\Sigma_\eta$  is the normal stacky fan to  $(Q_\eta, A_\eta)$ , we need only show that  $\overline{B}_\eta = \overline{A}_\eta \subset \mathbb{Z}^d \oplus \mathbb{Z}$ . By the definition of the Lafforgue fan,  $\tau \in \overline{B}_\eta$  if and only if it is minimal and constant on  $Q_i$  for some  $i$ . So the one cones of  $\overline{B}_\eta$  equal those of  $\overline{A}_\eta$ . But both sets consist of primitives of their one cones on vertical divisors, implying the equality.

To show that the pull back is isomorphic to  $(\mathcal{X}_\eta, \mathcal{Y}_{\iota_\eta(s)})$ , we prove any section of the form  $\iota_\eta(s)$  can be represented by a pullback of the universal section  $s_A$ . For this, we simply observe that the pullback of  $s_A$  to  $H^0(\mathcal{X}_\eta, \mathcal{O}_\eta(1))$  is  $\tilde{v}_\eta^*(s_A) = \sum_\alpha x_{(\alpha, \eta(\alpha))}$ . The group  $\mathbb{G}_{\Sigma(A)}$  acts transitively on the pullback of the space of very full sections of  $H^0(\mathcal{X}_\eta, \mathcal{O}_\eta(1))$  up to equivalence. Indeed, from the fundamental exact sequence for  $A$ , it is easy to see that there exists a  $\lambda \in \mathbb{G}_{\Sigma(A)}$  such that  $\tilde{v}_{\eta, \lambda}^*(s_A) = \sum_\alpha c_\alpha x_{(\alpha, \eta(\alpha))}$  for any  $\{c_\alpha\}$  satisfying  $\prod c_\alpha^{m_\alpha} = 1$  with  $\sum_\alpha m_\alpha \alpha = 0$ . Any full section has a representative in this class, yielding the claim.  $\square$

### 3. $\partial$ -FRAMED SYMPLECTOMORPHISMS

We begin this section by defining certain subgroups of symplectomorphism groups which we refer to as  $\partial$ -framed groups. The symplectic orbifolds we consider have boundary divisors that are preserved by the symplectomorphisms under consideration. Moreover, we would like to distinguish between subgroups that fix the boundary tangentially and those that do not. This aim would be easily achieved were our boundary divisor smooth and the symplectomorphisms fixed the boundary divisor pointwise. However, neither of these requirements are satisfied in our setting, so we must introduce a more elastic notion of framing than the usual one.

After defining the notion of a  $\partial$ -framed group, we proceed to examine the geometry of various symplectomorphisms contained in them. Up to Hamiltonian isotopy, the generators of our groups arise as monodromy maps around a singular symplectic orbifold. The type of permissible singularities that we will study fall into two classes. The first will be a toric degeneration of the symplectic orbifold into irreducible orbifolds glued along normal crossing divisors akin to the situation in complex geometry. The study of maximal degenerations of this type in the toric case was thoroughly analyzed in [1].

The second type of singularity we see is a stratified Morse singularity. This is studied in [29], but only the non-stratified case has been understood in the symplectic setting [54]. We will examine the general case and give a geometric description of the monodromy.

**3.1. Definitions.** Let  $(\mathcal{Y}, \omega)$  be a symplectic orbifold of real dimension  $2n$  with atlas  $\mathcal{U} = (U_\beta, G_\beta, \pi_\beta)_{\beta \in \mathcal{B}}$ . Most of the familiar constructions in symplectic geometry can be defined through the invariant manifold analogs in an atlas when working with symplectic orbifolds. For example, a Hamiltonian will mean a smooth function on  $\mathcal{Y}$  or, equivalently, a collection of smooth, compatible, invariant functions on  $(U_i, G_i)$ . Likewise, its flow can be computed in  $\mathcal{Y}$  or, for short time on a relatively compact subset, in each chart of the atlas. Types of submanifolds (Lagrangian, isotropic, symplectic), almost complex structures, Poisson brackets, symplectomorphisms are all defined locally and can be given precise meaning in the symplectic stack setting. We omit using the adjective “orbifold” for all of these terms throughout the paper. We leave the details of such definitions to the existing literature [3], [49], but will give details for structures that are less familiar.

We take  $\mathcal{J}$  to be the space of compatible almost complex structures on  $\mathcal{Y}$  and  $D = D_1 + \cdots + D_k$  a symplectic divisor, i.e. each  $D_i$  is a smooth symplectic suborbifold of real codimension 2. If there is an integrable  $J \in \mathcal{J}$  and  $\mathcal{Y}$  is a manifold, it makes sense to say that  $D$  is a divisor normal crossing singularities. We extend this to symplectic orbifolds in the following fashion. For every  $D_i$  and  $\beta \in \mathcal{B}$ , take  $D_i(\beta) = \pi_\beta^{-1}(D_i)$ .

**Definition 3.1.** Let  $J \in \mathcal{J}$ .

- (i) A symplectic divisor  $D$  will be called  $J$ -integrable if, for every  $D_i$  and every  $\beta \in \mathcal{B}$  there are symplectic neighborhoods  $V_i$  of  $D_i(\beta)$  such that  $J$  is integrable on  $V_i$  and  $D_i(\beta)$  is a complex divisor in  $V_i$  relative to  $J$ .
- (ii) A symplectic divisor  $D$  is a  $J$ -normal crossing divisor if, for every  $p \in \cap_{i \in I} D_i$ , there exists a  $J$ -holomorphic chart  $\psi : \cup_{i \in I} V_i \rightarrow U \subset \mathbb{C}^n$  near  $p \in V$  such that  $\psi(p) = 0$  and  $D \cap V = \psi^{-1}(\{(z_1, \dots, z_n) : z_{i_1} \cdots z_{i_k} = 0\})$ .
- (iii) A normal crossing divisor is  $J$ -standard if for every point  $p \in D$  above, there exists a  $J$ -holomorphic chart  $\psi$  such that  $\psi^* \omega = \omega_{st}$  where  $\omega_{st}$  denotes the standard symplectic form on  $U \subset \mathbb{C}^n$ .
- (iv) We say that a divisor is integrable, normal crossing or standard if there exists some  $J \in \mathcal{J}$  for which it is  $J$ -integrable,  $J$ -normal crossing or  $J$ -standard.

A consequence of having a  $J$ -standard normal crossing divisor is that the distance squared functions  $h_i : \mathcal{Y} \rightarrow \mathbb{R}$  from  $D_i$  (via the metric induced by  $\omega$  and  $J$ ) Poisson commute in neighborhoods of  $D$ . In other words, there exists an  $\varepsilon_J > 0$  for which  $\{h_i, h_j\} = 0$  on  $U_i \cap U_j$  where  $U_i = h_i^{-1}([0, \varepsilon_J])$ . We call any  $\varepsilon < \varepsilon_J$  commuting. For any commuting  $\varepsilon$ , we define  $\rho_i^\varepsilon = \lambda^\varepsilon \circ h_i$  where  $\lambda^\varepsilon : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a smooth monotonic function satisfying

$$(41) \quad \lambda^\varepsilon(r) = \begin{cases} r & r < \varepsilon/2 \\ \varepsilon & r \geq \varepsilon \end{cases}.$$

It is easy to see that  $\{\rho_i^\varepsilon, \rho_j^\varepsilon\} = 0$  on  $\mathcal{Y}$ . Given any  $\mathbf{t} \in \mathbb{R}^k$ , we define  $\tau(\mathbf{t})$  to be the flow of  $\sum_{i=1}^k t_i \rho_i^\varepsilon$ . The fact that the  $\rho_i$  Poisson commute implies that  $\tau(\mathbf{t}_1 + \mathbf{t}_2) = \tau(\mathbf{t}_1) \circ \tau(\mathbf{t}_2)$ . It is best to think of these maps as rotations, or twists, about the components of the divisor.

We take  $\text{Symp}(\mathcal{Y})$  to denote the topological group of symplectomorphisms with the  $C^\infty$ -topology and  $\text{Symp}_0(\mathcal{Y})$  the identity component.

For a Hermitian line bundle  $L$  over  $\mathcal{Y}$ , let  $\text{Symp}(L/\mathcal{Y})$  be the group of unitary line bundle automorphisms of  $L$  over symplectomorphisms of  $\mathcal{Y}$  and  $\text{Symp}_0(L/\mathcal{Y})$  its connected component (not to be confused with those maps of  $L$  lying over  $\text{Symp}_0(\mathcal{Y})$ ).

Given a standard normal crossing divisor  $D \subset \mathcal{Y}$  and a commuting  $\varepsilon > 0$ , we define  $\text{Symp}(\mathcal{Y}, D, \varepsilon)$  to consist of symplectomorphisms of  $\mathcal{Y}$  which preserve the distance to each  $D_i$  in a  $\varepsilon$  tubular neighborhood of  $D$ . Here we mean that for any  $\phi \in \text{Symp}(\mathcal{Y}, D, \varepsilon)$ , there exists a  $\varepsilon$  such that  $\phi^*(h_i|_{U_\varepsilon}) = h_i$  for all  $i$ . Equivalently, we can consider  $\text{Symp}(\mathcal{Y}, D, \varepsilon)$  to be the group of symplectomorphisms which commute with  $\tau(\mathbf{t})$  for every  $i$ . From this definition, it is clear the subgroup,

$$(42) \quad \mathbf{T}_\varepsilon := \{\tau(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^k\}$$

is contained in the center  $Z(\text{Symp}(\mathcal{Y}, D, \varepsilon))$ . It is an elementary exercise to show that the homotopy type of  $\text{Symp}(\mathcal{Y}, D, \varepsilon)$  is independent of commuting  $\varepsilon$ . Indeed,

using the symplectic neighborhood theorem and the condition of standard normal crossing, it is not difficult to see that each group is homotopy equivalent to the subgroup of symplectomorphisms of  $\mathcal{Y}$  which preserve each  $D_i$ . We will write this group as  $\text{Symp}(\mathcal{Y}, D)$ .

For a normal crossing divisor such as  $D \subset \mathcal{Y}$ , we write  $\text{Symp}(D)$  for the subgroup of  $\times_{i=1}^k \text{Symp}(D_i, D_i \cap (\cup_{j \neq i} D_j))$  consisting of  $\{\phi_i\}$  with  $\phi_i|_{D_i \cap D_j} = \phi_j|_{D_i \cap D_j}$ .

We call a collection of line bundles  $L = \{L_i\}$  where  $L_i$  is a line bundle over  $D_i$  compatible if  $L_i|_{D_i \cap D_j} \approx N_{D_i \cap D_j} D_j$ . A choice of such isomorphisms  $\mathbf{g} = \{\gamma_{i,j}\}$  will be called gluing data. Given such data, we take  $\text{Symp}_{\mathbf{g}}(L/D)$  to consist of symplectic line bundle automorphisms  $\{\psi_i\}$  of  $L_i/D_i$  which lie over some  $\{\phi_i\} \in \text{Symp}(D)$  and for which  $d\phi_j|_{D_j \cap D_i} = \gamma_{i,j}(\psi_i)$ . We will simply write  $\text{Symp}(L/D)$  when the gluing data is evident. For example, in our context of a normal crossing divisor  $D \subset \mathcal{Y}$ , we take  $N_D \mathcal{Y}$  for the collection of normal bundles  $\{N_{D_i} \mathcal{Y}\}$  with the induced gluing data.

**Definition 3.2.** Given  $\mathcal{Y}$  with a standard normal crossing divisor  $D$ , we say that a compactly generated, closed subgroup  $\mathbf{F} \subseteq \text{Symp}(N_D \mathcal{Y}/D)$  is a  $\partial$ -frame group of  $(\mathcal{Y}, D)$ .

Let  $j : D \rightarrow \mathcal{Y}$  be the inclusion map and  $j^\# : \text{Symp}(\mathcal{Y}, D) \rightarrow \text{Symp}(N_D \mathcal{Y}/D)$  the restriction of the derivative. Given a  $\partial$ -frame group  $\mathbf{F}$ , we will say  $\phi \in \text{Symp}(\mathcal{Y}, D)$  is an  $\mathbf{F}$ -framed, or framed, symplectomorphism if  $j^\#(\phi) \in \mathbf{F}$ . Denote the group of  $\mathbf{F}$ -framed symplectomorphisms by  $\text{Symp}^{\mathbf{F}}(\mathcal{Y}, D)$ .

It may be the case that symplectomorphisms of  $N_D \mathcal{Y}$  do not extend to those on  $\mathcal{Y}$ . Including such maps into the  $\partial$ -frame group has no effect on the framed symplectomorphism group. To take care of this redundancy, we define a reduced framing as follows.

**Definition 3.3.** A  $\partial$ -frame group  $\mathbf{F}$  will be called reduced if for every  $\phi \in \mathbf{F}$  there exists a  $\psi \in \text{Symp}(\mathcal{Y}, D)$  such that  $j^\#(\psi) = \phi$ . The maximal subgroup  $\mathbf{F}^{\text{red}}$  of any  $\partial$ -frame group  $\mathbf{F}$  consisting of such  $\phi$  will be called the  $(\mathcal{Y}, D)$  reduction of  $\mathbf{F}$ .

Of course, the closure of the image  $j^\#(\mathbf{G})$  of any subgroup  $\mathbf{G} \subset \text{Symp}(\mathcal{Y}, D)$  is a reduced  $\partial$ -frame group. An important class of these groups occurs in the following definition:

**Definition 3.4.** A  $\partial$ -gauge group is a  $\partial$ -frame group contained in  $j^\#(\mathbf{T}_\varepsilon)$ .

The motivation for defining  $\partial$ -gauge groups stems from the desire to exert control over a group similar to the group  $(S^1)^k$  of complex multiplications on  $\times_{i=1}^k N_{D_i} \mathcal{Y}$ . Such a group would keep track of rotations around the boundary divisor  $D$  of  $\mathcal{Y}$ . Unfortunately, for  $\dim \mathcal{Y} > 2$ , this group is not contained in  $\text{Symp}(N_D \mathcal{Y}/D)$  as the compatibility condition is violated. The  $\partial$ -gauge group and its subgroups can be thought of as an approximation to such a rotation group.

One of the central points of  $\partial$ -frame groups is to allow more flexibility than fixing the boundary and a normal bundle on it. In fact, this more restrictive case occurs as the framed group  $\text{Symp}^1(\mathcal{Y}, D)$  with the trivial framing  $\mathbf{1} = \{1\}$ . This fits nicely into the more general framework as follows.

**Proposition 3.5.** For any reduced  $\partial$ -frame group  $\mathbf{F}$ , the map:

$$(43) \quad \text{Symp}^{\mathbf{F}}(\mathcal{Y}, D) \xrightarrow{j^\#} \mathbf{F}$$

defines a topological fiber bundle with fiber  $\text{Symp}^1(\mathcal{Y}, D)$ .

*Proof.* It follows from the definition of reduced framings that  $j^\#$  is the quotient of  $\mathrm{Symp}^{\mathbf{F}}(\mathcal{Y}, D)$  by the closed, normal subgroup  $\mathrm{Symp}^1(\mathcal{Y}, D)$ . Thus, to prove the claim, one need only show the existence of a local section around the identity which is an exercise in elementary symplectic geometry.  $\square$

This gives an important corollary.

**Corollary 3.6.** *Suppose  $\mathbf{F}_1 \subseteq \mathbf{F}_2$  are reduced  $\partial$ -frame groups. Then there is a homotopy fiber sequence:*

$$(44) \quad \mathrm{Symp}^{\mathbf{F}_1}(\mathcal{Y}, D) \rightarrow \mathrm{Symp}^{\mathbf{F}_2}(\mathcal{Y}, D) \rightarrow \frac{\mathbf{F}_2}{\mathbf{F}_1}$$

For a reduced  $\partial$ -frame group  $\mathbf{F}$ , we define  $\mathbf{F}^{rel}$  to be the group generated by  $\mathbf{F}$  and  $\mathbf{T}$ . As all elements in  $\mathrm{Symp}(\mathcal{Y}, D, \varepsilon)$  are required to commute with  $\tau(\mathbf{t})$  and  $\mathbf{F}$  is reduced,  $\mathbf{F}^{rel}$  is a central extension of  $\mathbf{F}$ . The following proposition is then an elementary application of corollary 3.6.

**Proposition 3.7.** *For any reduced  $\partial$ -frame group  $\mathbf{F}$ , there is a homotopy exact sequence*

$$(45) \quad \mathrm{Symp}^{\mathbf{F}}(\mathcal{Y}, D) \rightarrow \mathrm{Symp}^{\mathbf{F}^{rel}}(\mathcal{Y}, D) \rightarrow (S^1)^t.$$

*Proof.* We need only observe that  $\mathbf{F}^{rel}$  is a split finite dimensional central extension of  $\mathbf{F}$ . Since  $\mathbf{F} \cap \mathbf{T}$  is closed in  $\mathbf{F}^{rel}$ , we have that  $\mathbf{F}^{rel}/\mathbf{F} \approx \mathbf{T}/\mathbf{T} \cap \mathbf{F} \approx \mathbb{R}^l \times (S^1)^t$ . So by corollary 3.6 we have the result.  $\square$

Our primary examples of symplectic orbifolds arise as toric hypersurfaces of  $\mathcal{X}_Q$ . Generally,  $Q$  is not assumed to be a simple polytope and so the hypersurfaces will generally be singular along a complex codimension 2 subspace  $\mathcal{Y}_{sing} \subset D$  of  $\mathcal{Y}$ . To deal with these cases, we extend our notion of  $\partial$ -framing.

Assume that  $D$  is a  $J$ -integrable divisor and let  $\mathcal{R}(\mathcal{Y}, D)$  be a collection  $\mathcal{R} = \{\phi_\varepsilon : (\tilde{\mathcal{Y}}, \tilde{D}) \rightarrow (\mathcal{Y}, D)\}$  of normal crossing resolutions of  $(\mathcal{Y}, D)$ . We assume each  $(\tilde{\mathcal{Y}}, \tilde{D})$  is a smooth symplectic orbifold with  $\tilde{J}$ -standard normal crossing divisors,  $\phi_\varepsilon^*(\omega) = \tilde{\omega}$  off of an  $\varepsilon$  neighborhood of  $\mathcal{Y}_{sing}$  and  $\phi_\varepsilon$  is  $(\tilde{J}, J)$ -holomorphic in a neighborhood of  $\tilde{D}$ . The ability to resolve singularities in this setting follows from the integrability condition of  $D$  and Hironaka. However, it is not clear that one may generally force the divisor to be standard in the symplectic sense defined above. To insure that there exists a non-empty  $\mathcal{R}$ , we define:

**Definition 3.8.** We say that  $(\mathcal{Y}, D)$  is a standard symplectic stack if there exists a nonempty collection  $\mathcal{R}$ . We call  $\mathcal{R}$  a resolving collection of  $(\mathcal{Y}, D)$ .

If  $(\mathcal{Y}, D)$  is standard, then  $\mathrm{Symp}_{\mathcal{R}}(\mathcal{Y}, D)$  is defined to consist of all symplectomorphisms  $\psi \in \mathrm{Symp}(\mathcal{Y} - \mathcal{Y}_{sing}, D - \mathcal{Y}_{sing})$  that lift to a symplectomorphism  $\tilde{\psi} \in \mathrm{Symp}(\tilde{\mathcal{Y}}, \tilde{D})$  for all  $(\tilde{\mathcal{Y}}, \tilde{D}) \in \mathcal{R}(\mathcal{Y}, D)$ . We note that this is the coarsest group that could be defined relative to  $\mathcal{R}$ , ignoring any of the subtleties of the combinatorics of the distinct resolutions in  $\mathcal{R}$ . Likewise, a  $\partial$ -frame group  $\mathbf{F}$  for a standard symplectic stack  $(\mathcal{Y}, D)$  is a subgroup of  $\mathrm{Symp}(N_{D-\mathcal{Y}_{sing}}(\mathcal{Y} - \mathcal{Y}_{sing}/D - \mathcal{Y}_{sing}))$  that has a lift to  $\mathrm{Symp}(N_{\tilde{D}}\tilde{\mathcal{Y}}, \tilde{D})$  for every  $(\tilde{\mathcal{Y}}, \tilde{D})$ . The definition of the framed group  $\mathrm{Symp}^{\mathbf{F}}(\mathcal{Y}, D)$  and the results above all hold in this case for obvious reasons.

Our primary example of standard symplectic stacks arise in the toric setting.

**Proposition 3.9.** *Suppose  $(\mathcal{X}, \partial\mathcal{X})$  is a Kähler DM toric stack, where  $\partial\mathcal{X}$  is the toric boundary. For generic complete intersections  $\mathcal{Y} \subset \mathcal{X}$ , taking  $D = \partial\mathcal{X} \cap \mathcal{Y}$ , the pair  $(\mathcal{Y}, D)$  is a standard symplectic stack.*

**3.2. Toric degeneration monodromy.** In this subsection, we obtain the local model for monodromy around a toric hypersurface degeneration. Assume  $(\mathcal{X}, \omega)$  is a symplectic orbifold of dimension  $n$  with  $r$ -dimensional Hamiltonian torus action. We write  $\mathbb{T}^r$  for the torus,  $\mathfrak{t}_r$  (or  $\mathfrak{t}$ ) for its Lie algebra and  $v_- : \mathfrak{t} \rightarrow \text{Vect}(\mathcal{X})$  the infinitesimal action. Let  $J$  be a compatible almost complex structure on  $\mathcal{X}$  which is invariant with respect to the action and  $\mu : \mathcal{X} \rightarrow \mathfrak{t}^\vee$  the moment map.

We define the map

$$(46) \quad \kappa : \mu(\mathcal{X}) \rightarrow \text{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathfrak{t})$$

by taking  $\kappa_u(v) = d\mu_p(Jv_v)$  where  $\mu(p) = u$ . Note that this is well defined only under the assumption that  $J$  is  $\mathbb{T}^r$  invariant. Alternatively, we may think of the map  $\kappa$  as giving the metric restricted to the infinitesimal action vector fields  $g|_{\mathfrak{t}} \in \mathfrak{t}^\vee \otimes \mathfrak{t}^\vee$ . Given two vectors  $v, w \in \mathfrak{t}$ , we will write

$$(47) \quad \langle v, w \rangle_{\kappa_u} := [\kappa_u(v)](w) = g_p(v_v, v_w).$$

Suppose we have the commutative diagram

$$(48) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\mu} & \mathfrak{t}^\vee \\ \downarrow F & & \downarrow f \\ \mathbb{C} & \xrightarrow{\mu_{\mathbb{C}}} & \mathbb{R} \end{array}$$

where  $F$  is holomorphic and  $\mu_{\mathbb{C}} = |\cdot|^2$ .

We assume that  $F$  has no critical values outside 0 and let  $\mathcal{X}^\circ = \mathcal{X} - F^{-1}(0)$ . Recall that  $\omega$  defines a Hamiltonian connection on the smooth map  $F : E \rightarrow B$  by taking the horizontal distribution to be the symplectic orthogonal to the tangent space of the fiber. As usual, this allows us to lift any vector field on  $\mathbb{C}^*$  to  $\mathcal{X}^\circ$  via

$$(49) \quad \xi : \text{Vect}(\mathbb{C}^*) \rightarrow \text{Vect}(\mathcal{X}^\circ).$$

Recall that the map  $\mu_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{R}$  is the moment map for complex multiplication by  $e^{-2it}$ . We take  $\rho = -2iz\partial_z$  to denote its infinitesimal vector field on  $B$ . Note also that derivative of  $f$  at a point  $p$  gives a natural function  $df : \mathfrak{t}^\vee \rightarrow \mathfrak{t}$ .

**Lemma 3.10.** *Let  $p \in \mathcal{X}^\circ$  and  $q = \mu(p)$ . The horizontal lift  $\xi(\rho)$  of  $\rho$  at  $p$  is dependent only on  $q$  and is  $v_{\delta_q}$  where  $\delta_q \in \mathfrak{t}$  is given:*

$$(50) \quad \delta_q = \frac{4f(q)}{\|df_q\|_{\kappa_q}^2} df_q$$

*Proof.* Let  $p' = F(p)$  and  $Y \in T_p E$ . Observe, by the definition of the moment map and the commutative diagram 48,

$$\begin{aligned} \omega(v_{df_q}, Y) &= \langle d\mu(Y), df_{\mu(p)} \rangle, \\ &= d(f \circ \mu)_p(Y), \\ &= d(\mu_{\mathbb{C}} \circ F)_p(Y). \end{aligned}$$

In particular, we see that  $v_{df_q}(p) \in (T_p F^{-1}(p'))^{\perp_\omega}$  and that  $d\mu_{\mathbb{C}}[dF(v_{df_{\mu(p)}})] = 0$ . The latter equality gives us that  $\rho \wedge dF(v_{df_q}) = 0$  so that  $v_{df_q}$  is a real multiple of

$\xi(\rho_{p'})$ . To evaluate this constant, let  $v_{df_q} = \gamma_p$  and define  $r_p$  as

$$(51) \quad dF(\gamma_p) = r_p \rho_{p'}.$$

We take inner product with  $\rho$  on both sides so that

$$\begin{aligned} r_p &= \frac{\langle dF(\gamma_p), \rho_{p'} \rangle}{\langle \rho_{p'}, \rho_{p'} \rangle}, \\ &= \frac{d\mu_{\mathbb{C}}(JdF(\gamma_p))(1)}{4\mu(F(p))}, \\ &= \frac{d(\mu_{\mathbb{C}} \circ F)(J\gamma_p)}{4f(\mu(p))}, \\ &= \frac{df_{\mu(p)}(d\mu_q(Jv_{df_q}))}{4f(q)}, \\ &= \frac{\langle df_q, df_q \rangle_{\kappa_q}}{4f(q)}. \end{aligned}$$

Letting  $\delta_q = r_p^{-1} df_q$  then gives  $dF(v_{\delta_q}) = \rho_{p'}$ , yielding the claim.  $\square$

Given any smooth function  $\tilde{f} : \mu(\mathcal{X}) \rightarrow \mathfrak{t}$ , the vector field  $v_{\tilde{f}(\mu(p))}(p)$  is easily integrated to  $\phi_t^{\tilde{f}} : \mathcal{X} \rightarrow \mathcal{X}$  where  $\phi_t^{\tilde{f}}(p) = \exp(t\tilde{f}(\mu(p))) \cdot p$ . Thus the previous lemma gives an explicit description of the symplectic monodromy map of  $F$ . Namely, take  $\tilde{f} = \left( \frac{4f(q)}{\|df_q\|_{\kappa_q}^2} \right) df_q$ , and for any  $\varepsilon > 0$  the monodromy map is

$$(52) \quad \phi_1^{\tilde{f}} : F^{-1}(\varepsilon) \rightarrow F^{-1}(\varepsilon).$$

We now define the local model for a degeneration as a toric hypersurface degeneration. To study the monodromy around such a degeneration, we first examine the monodromy with respect to the ambient toric variety and then adjust this map slightly near the critical points of the degeneration to obtain a characterization of the monodromy on the hypersurface.

Recall from 2.2 that  $A \subset \Lambda$  gives a subset of equivariant linear sections of a line bundle  $\mathcal{L}_A$  on a toric stack  $\mathcal{X}_Q$  specified by  $(Q, A)$ . If  $S = \{(Q_i, A_i)\}_{i \in I}$  is a regular subdivision of  $(Q, A)$  and  $\eta : Q \rightarrow \mathbb{R}$  is an integral defining function of  $S$ , then we defined a toric hypersurface degeneration  $(\mathcal{X}_\eta, \mathcal{Y}_s)$  of  $(\mathcal{X}_Q, \mathcal{Y}_s)$ .

To perform symplectic parallel transport around the critical value of a degenerating family, we choose the standard symplectic form  $\omega$  on  $\mathcal{X}_\eta$ . The moment map  $\mu_\eta : \mathcal{X}_\eta \rightarrow \Lambda_{\mathbb{R}} \oplus \mathbb{R}$  can be found using techniques from subsection 2.1. The diagram 18 for the polyhedra  $Q_\eta$  is

$$(53) \quad \begin{array}{ccccc} \mathbb{C}^{\overline{Q}_\eta} & \xleftarrow{\text{inc}} & \mu_H^{-1}(\omega) & \xrightarrow{\rho_{A_\eta}} & \mathcal{X}_\eta & \xlongequal{\quad} & \mathcal{X}_\eta \\ & \downarrow \mu_{\overline{Q}_\eta} & \downarrow \mu_{\overline{Q}_\eta} & & & & \downarrow \mu_\eta \\ \mathbb{R}^{\overline{Q}_\eta} & \xlongequal{\quad} & \mathbb{R}^{\overline{Q}_\eta} & \xleftarrow{\beta_{\overline{Q}_\eta}^\vee} & \Lambda_{\mathbb{R}} \oplus \mathbb{R} & \xleftarrow{\text{inc}} & Q_\eta. \end{array}$$



We recall that a degenerating family  $F_\eta : \mathcal{X}_\eta \rightarrow \mathbb{C}$  associated to  $\eta$  is defined by the diagram

$$(54) \quad \begin{array}{ccc} \mathcal{X}_\eta & \xleftarrow{\rho_H} & \mu_H^{-1}(\omega) \\ F_\eta \downarrow & & \downarrow \text{inc} \\ \mathbb{C} & \xleftarrow{\tilde{F}_\eta} & \mathbb{C}^{\overline{Q}_\eta} \end{array}$$

where  $\tilde{F}_\eta$  is given by the monomial  $\prod_{i \in \overline{Q}_\eta^v} z_i^{c_{\eta,i}}$ . We recall that  $c_{\eta,i}$  is the denominator of  $d\varsigma_i$  where  $\varsigma_i = \beta_{Q_\eta}(i)$ . One can fill in diagram 48 as

$$(55) \quad \begin{array}{ccc} \mathbb{C}^{\overline{Q}_\eta} & \xrightarrow{\mu_{\overline{Q}_\eta}} & \mathbb{R}^{\overline{Q}_\eta} \\ \downarrow \tilde{F}_\eta & & \downarrow \tilde{f}_\eta \\ \mathbb{C} & \xrightarrow{\mu_{\mathbb{C}}} & \mathbb{R} \end{array}$$

where  $\tilde{f}_\eta(r_1, \dots, r_{|\overline{Q}_\eta|}) = \prod_{i \in \overline{Q}_\eta^v} r_i^{c_{\eta,i}}$ . Letting  $Y = \mu_{L_{\overline{Q}_\eta}}^{-1}(\omega)$ , we assemble the commutative diagrams 53, 54 and 55 into the following diagram which defines  $f_\eta$ :

$$\begin{array}{ccccc} \mathcal{X}_\eta & \xrightarrow{\mu_\eta} & \Lambda_{\mathbb{R}} \oplus \mathbb{R} & & \\ & \swarrow \rho_\eta & \searrow \beta_{\overline{Q}_\eta}^\vee & & \\ & Y & \longrightarrow & \mathbb{R}^{\overline{Q}_\eta} & \\ & \downarrow \text{inc} & & \downarrow = & \\ & \mathbb{C}^{\overline{Q}_\eta} & \xrightarrow{\mu_{\overline{Q}_\eta}} & \mathbb{R}^{\overline{Q}_\eta} & \\ & \swarrow \tilde{F}_\eta & & \searrow \tilde{f}_\eta & \\ \mathbb{C} & \xrightarrow{\mu_{\mathbb{C}}} & \mathbb{R} & & \\ & & & & \downarrow f_\eta \end{array}$$

Recall from 2.2 that  $\eta$  defines an affine function  $\varsigma_i$  on  $\Lambda$  for each  $i \in I$  of the regular subdivision  $S = \{(Q_i, A_i) : i \in I\}$ . The following proposition gives an explicit characterization of  $f_\eta$  as well as the monodromy of  $F_\eta$  around 0.

**Proposition 3.11.** *Let  $(\mathbf{r}, t)$  be coordinates for  $\Lambda_{\mathbb{R}} \oplus \mathbb{R}$ . Then  $f_\eta$  can be written as*

$$(56) \quad f_\eta(\mathbf{r}, t) = \prod_{i \in I} [c_{\eta,i}(\varsigma_i(\mathbf{r}) - t)]^{c_{\eta,i}}.$$

The normalized derivative  $\frac{4f(\mathbf{r}, t)}{\|df_{(\mathbf{r}, t)}\|_{\kappa(\mathbf{r}, t)}^2} df_{(\mathbf{r}, t)}$  converges uniformly to  $c_{\eta,i}^{-1}(d\varsigma_i - dt)$  on compactly supported subsets of the interior of  $Q_{\eta,i}$ .

*Proof.* For the first part of the claim, we re-examine the map  $\beta_{\overline{Q}_\eta}^\vee$ . Recall that the definition of  $\beta_{\overline{Q}_\eta}^\vee : \mathbb{Z}^{\overline{Q}_\eta} \rightarrow \Lambda^\vee \oplus \mathbb{Z}^\vee$  gave  $\beta_{\overline{Q}_\eta}^\vee(e_b)$  as a minimal supporting hyperplane for the facet  $b$  of  $Q_\eta$ . Now, for every  $i \in I$ , we write  $\varsigma_i$  as the sum  $L_i - m_i$  of a linear function  $L_i \in \Lambda_Q^\vee$  and a constant  $m_i$ . Notice that  $\varsigma_i(\alpha) \geq \eta(\alpha)$  for all  $\alpha \in A$  with equality iff  $\alpha \in A_i$ . Let  $f^\vee = (0, 1)^\vee \in \Lambda^\vee \oplus \mathbb{Z}^\vee$  and observe that  $(L_i - f^\vee)(\alpha, \eta(\alpha)) \geq m_i$  for all  $\alpha \in A$  with equality on the supporting hyperplane

$Q_i$ . We see then that  $\beta_{\overline{Q}}(e_i) = c_{\eta,i}(L_i - f^\vee)$  and that the order of  $D$  along  $D_i$  is  $n_i = c_{\eta,i}m_i$ . This implies that  $\gamma = -\sum_{i \in I} c_{\eta,i}m_i e_i - \sum_{j \in \overline{Q}_\eta} n_j e_j$  yields  $D_{Q_\eta}$  so, for any  $i \in I$ , the composition

$$\begin{aligned} e_i^\vee \circ \beta_{\overline{Q}_\eta}^\vee &= e_i^\vee \circ \beta_{\overline{Q}}^\vee + e_i^\vee(\gamma), \\ &= \beta_{\overline{Q}}^\vee(e_i) - c_{\eta,i}m_i, \\ &= c_{\eta,i}[(L_i - m_i) - f^\vee], \\ &= c_{\eta,i}(\varsigma_i - f^\vee). \end{aligned}$$

But the function  $r_i : \mathbb{R}^{\overline{Q}_\eta} \rightarrow \mathbb{R}$  is induced from  $e_i^\vee$  so that the composition  $e_i^\vee \circ \beta_{\overline{Q}_\eta}^\vee$  above occurs as one monomial in the expression for  $f_\eta$ .

We use this and the concavity of  $\tilde{\eta}$  for the second claim. Before proving this though, we observe that, letting  $\mathbf{R} = (r_1, \dots, r_{|\overline{Q}_\eta|}) \in \mathbb{R}^{\overline{Q}_\eta}$ , the map  $\tilde{\kappa}$  for  $\mathbb{C}^{\overline{Q}_\eta}$  can easily be derived to be

$$(57) \quad \tilde{\kappa}_{\mathbf{R}} = \begin{bmatrix} 4r_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 4r_{|\overline{Q}_\eta|} \end{bmatrix}.$$

This restricts to the map  $\kappa$  on  $\Lambda_{\mathbb{R}} \oplus \mathbb{R}$  after the substitution  $r_i = c_{\eta}(\varsigma_i - t)$ . We now calculate

$$(58) \quad df_{\tilde{\eta}}(\mathbf{R}) = \tilde{f}_\eta(\mathbf{R}) \cdot \sum_{i \in I} \frac{c_{\eta,i}}{r_i} dr_i.$$

Then utilizing the normalization factor, the formula above for  $\kappa$  and computing away from  $Q_{\eta,i}$  we have:

$$\begin{aligned} \frac{4f(\mathbf{r}, t)}{\|df_{\mathbf{r}, t}\|_{\kappa(\mathbf{r}, t)}^2} df_{\mathbf{r}, t} &= \frac{4}{\left\| \sum_{i \in I} \frac{c_{\eta,i}}{\varsigma_i - t} d(\varsigma_i - t) \right\|_{\kappa(\mathbf{r}, t)}^2} \sum_{i \in I} \frac{c_{\eta,i}}{\varsigma_i - t} d(\varsigma_i - t), \\ &= \frac{1}{\sum_{i \in I} \frac{c_{\eta,i}^2}{\varsigma_i - t}} \sum_{i \in I} \frac{c_{\eta,i}}{\varsigma_i - t} d(\varsigma_i - t). \end{aligned}$$

From the observations above we have  $(\varsigma_i - t)|_{Q_{\eta,i}} = 0$  and is positive on the rest of  $Q_\eta$ . In particular, for any relatively compact open subset  $U_i \subset Q_{\eta,i}$  off the intersection  $(\cup_{j \neq i} Q_{\eta,j}) \cap Q_{\eta,i}$ , we may set  $(\varsigma_j - t)(\mathbf{r}, t) = s_j > C > 0$  for all  $j \neq i$ ,  $(\mathbf{r}, t) \in U_i$ . Taking  $(\mathbf{r}, t)$  to be in a normal neighborhood of  $U_i$ , we then have

$$\begin{aligned} \lim_{\varsigma_i \rightarrow t} \frac{4f(\mathbf{r}, t)}{\|df_{\mathbf{r}, t}\|_{\kappa(\mathbf{r}, t)}^2} df_{\mathbf{r}, t} &\simeq \lim_{\varsigma_i \rightarrow t} \frac{c_{\eta,i}d(\varsigma_i - t) + \sum_{j \neq i} c_{\eta,j}s_j^{-1}(\varsigma_i - t)d(\varsigma_j - t)}{c_{\eta,i}^2 + \sum_{j \neq i} c_{\eta,j}^2 s_j^{-1}(\varsigma_i - t)}, \\ &= c_{\eta,i}^{-1}d(\varsigma_i - t). \end{aligned}$$

□

We now take a moment to explain the meaning of this proposition. Combined with lemma 3.10, the first part of the proposition gives an explicit formula for monodromy of  $\mathcal{X}_Q$  about a toric degeneration. In the second claim we obtain the qualitative behavior of this monodromy with respect to the degenerated components. Namely, on the component  $\mathcal{X}_{Q_i}$  of the degeneration, we have that the

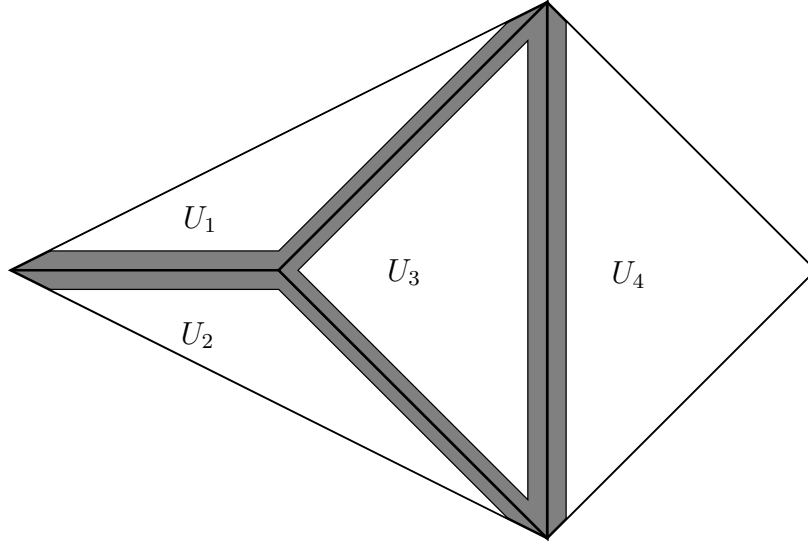


FIGURE 2. Regions of finite order monodromy

limiting transformation of the monodromy map equals a toric map (i.e. multiplication by an element of the isotropy group of  $(\mathcal{X}_{Q_i}, \mathcal{Y}_s)$  for any given  $s$ ). Thus we find that the symplectic operation of parallel transport, which is very far from being holomorphic, limits to a holomorphic map on the degeneration.

We write these observations as a corollary.

**Corollary 3.12.** *For a regular subdivision  $\eta$  and any sufficiently small  $\varepsilon > 0$ , there is an induced decomposition  $\mathcal{X}_Q = \cup_{i \in I} U_i$  such that degeneration monodromy relative to  $\eta$  is the convolution of toric multiplication  $\exp(c_{\eta,i}^{-1} d\eta_{Q_i})$  on each  $U_i$  along  $\varepsilon$  neighborhoods of their intersections.*

To obtain the structure of parallel transport on the hypersurface, we simply define an appropriate perturbation of these maps which preserve the hypersurface. This is a less elegant approach than the straightening lemma of [1], but one which works for arbitrary toric degenerations and yields a description that is Hamiltonian isotopic in the maximal case. We only need to assume that the defining section  $s$  is in the complement of the  $A$ -determinant. Since our hypersurfaces are fixed in the degenerate fiber by the monodromy map, the ambient toric monodromy approximates the hypersurface map up to a negligible factor.

To describe the hypersurface monodromy map, let  $\mathcal{Z}_\eta(0) = \cup_{i \in I} \mathcal{Z}_i(0)$  be the components of the degenerated hypersurface and  $g_i : \mathcal{Z}_i(t) \rightarrow \mathcal{Z}_i(t)$  the Kähler automorphism corresponding to  $\exp(c_{\eta,i}^{-1} d\eta_{Q_i})$ .

**Proposition 3.13.** *Assume  $E_A(s') \neq 0$ . There exists a decomposition of  $\mathcal{Z}_\eta(t) = \cup_{i \in I} \overline{V_i}$  such that  $V_i \approx \mathcal{Z}_i(0) - \partial \mathcal{Z}_i(0)$  and the monodromy map  $\phi_\eta : \mathcal{Z}_\eta(t) \rightarrow \mathcal{Z}_\eta(t)$  equals  $g_i$  on  $V_i$  off a  $\varepsilon$  neighborhood  $V_i(\varepsilon)$  of  $\partial V_i$ , and is interpolated smoothly over  $V_i(\varepsilon)$  by a Hamiltonian flow.*

**3.3. Stratified Morse singularities.** In [54], it was seen that symplectic monodromy around a Morse singularity has infinite order in the symplectic mapping

class group for any dimension. In this paper, these types of singularities are encountered as a nondegenerate case. For the degenerate case, we need a different model whose critical fiber is in fact smooth, but fails to transversely intersect the boundary divisor. Restricting to the boundary divisor, we see a Morse singularity and expect that the monodromy on the ambient space extends the monodromy of the restriction.

We have one essential obstruction to pursuing this naively. Namely, if our parallel transport map preserves a boundary divisor  $\mathcal{D}$  in  $\mathcal{Y}$ , then  $\mathcal{D}$  must be horizontal relative to the symplectic orthogonal connection. On the other hand, if a smooth fiber does not intersect  $\mathcal{D}$  transversely at  $p$ , then the symplectic orthogonal will be normal, or vertical, to  $\mathcal{D}$ . This holds for all symplectic connections in  $\Omega^2(\mathcal{Y})$ . We resolve this difficulty by considering a singular connection on  $\mathcal{D}$  and show that the parallel transport vector field extends over the singularities and preserves the symplectic form of the fiber up to a negligible factor.

Let  $U \subset \mathbb{C}^n$  to be a neighborhood of zero,  $D(m) = \{z_1 z_2 \cdots z_m = 0\} \cap U$  and  $D_{[m]} = \cap_{i=1}^m D_i \cap U$ . Given a linear function  $L : \mathbb{C}^m \rightarrow \mathbb{C}$  with nonzero restrictions to each coordinate line, we let  $f_L : U \rightarrow \mathbb{C}$  be the function

$$(59) \quad f_L(z_1, \dots, z_n) = L(z_1, \dots, z_m) + \frac{1}{2} (z_{m+1}^2 + \cdots + z_n^2).$$

While this map is smooth, the fiber over zero does not transversely intersect the divisor  $D(m)$  along  $D_{[m]}$  at 0. We call such a point a degenerate point and its value a degenerate value.

**Definition 3.14.** Let  $\mathcal{Y}$  be a symplectic orbifold with normal crossing divisor  $D = \cup_{i=1}^m D_i$  and  $p \in \cap_{i=1}^m D_i$ . We say that a function  $f : \mathcal{Y} \rightarrow \mathbb{C}$  is a stratified Morse function at  $p$  relative to  $D$  if there exists a holomorphic chart  $\phi : (U, D(m)) \rightarrow (\mathcal{Y}, D)$  centered at  $p$  and a linear function  $L : \mathbb{C}^m \rightarrow \mathbb{C}$  such that  $f \circ \phi = f_L + f(p)$ . In this case we say that  $f$  is stratified with codimension  $m$ ,  $p$  is a degenerate point of  $f$  and that  $f(p)$  is a degenerate value of  $f$ .

In the non-stratified setting we have a useful criteria for deciding when a function is Morse. A similar tool, whose proof is straightforward, is available in the stratified case.

**Proposition 3.15.** *Let  $f : U \rightarrow \mathbb{C}$  be a holomorphic function. Then  $f$  is a stratified Morse function at 0 relative to  $D$  if and only if i)  $df_0 \neq 0$  on any coordinate space properly containing  $D_{[m]}$ , ii)  $df_0(T_0 D_{[m]}) = 0$ , and iii)  $\text{Hess}_0(f)$  is non-degenerate on  $T_0 D_{[m]}$ .*

Let  $D$  be a  $\varepsilon$  radius disc about the origin in  $\mathbb{C}$  and  $\tilde{U} = f_L^{-1}(D) \subset \mathbb{C}$ . As was pointed out above, the symplectic orthogonal connection on  $f_L : \tilde{U} \rightarrow D$  needs to be corrected in order to preserve the boundary  $D(m)$ . We implement a form of Moser's trick by integrating a path of equivalent symplectic forms, perform parallel transport relative to the corrected form and then flow back to the standard form.

Let  $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a convex function which satisfies

$$(60) \quad \rho_\varepsilon(r) = \begin{cases} \varepsilon^2 r & \text{for } r < \varepsilon \\ r^2 & \text{for } r > 2\varepsilon \end{cases}.$$

Then define  $\omega_\varepsilon$  on  $\mathbb{C}^n$  to be the symplectic form obtained by the Kähler potential

$$(61) \quad p_\varepsilon(z_1, \dots, z_n) = \sum_{i=1}^m \rho_\varepsilon(|z_i|) + \sum_{i=1}^{m+1} |z_i|^2.$$

It is clear that  $\omega_\varepsilon$  is a smooth symplectic form away from  $D(m)$  and singular on  $D(m)$ . While singular, an elementary application of Stokes theorem shows that the integral  $\int_D \omega_\varepsilon$  is constant on any closed disc with boundary outside an  $\varepsilon$  neighborhood of  $D(m)$ . Let  $X_\varepsilon$  be the vector field which is the fiberwise dual of the 1-forms  $d^c(p - p_\varepsilon)$  with respect to  $f_L$ . Then  $X_\varepsilon$  may be integrated on all of  $\tilde{U}$ . Indeed, a quick check shows that the vector field defined by  $X_\varepsilon$  is locally Lipschitz on  $\tilde{U} - D(m)$ . We denote by  $\Phi_t : \tilde{U} \rightarrow \tilde{U}$  the singular symplectomorphism obtained through the integration.

**Definition 3.16.** For any stratified Morse function  $f : \mathcal{Y} \rightarrow \mathbb{C}$ , we will call conjugation of parallel transport around 0 relative to  $\omega_1$  by  $\Phi_1$  modified symplectic parallel transport.

We now investigate the local behavior of modified symplectic parallel transport for  $f_L : \mathbb{C}^n \rightarrow \mathbb{C}$ . As the computation is local, we may extend the symplectic form  $\omega_1$  over all of  $\mathbb{C}^n$  as

$$(62) \quad \omega = \frac{i}{2} \left( \sum_{i=1}^m \frac{dz_i \wedge d\bar{z}_i}{|z_i|} + \sum_{i=m+1}^n dz_i \wedge d\bar{z}_i \right).$$

Let  $L(z_1, \dots, z_m) = c_1 z_1 + \dots + c_m z_m$  and note that we may change coordinates by multiplying  $z_i$  with  $e^{-\text{Arg}(c_i)i}$  without affecting the map  $\Phi$ , so that we may assume  $c_i \in \mathbb{R}_+$ . Giving  $\mathbb{C}$  the coordinate  $w$ , it is easy to compute that the horizontal lift of the vector field  $\partial_w$  at  $z = (z_1, \dots, z_n) \in U - D(m)$  is

$$(63) \quad \xi(z) = \frac{1}{c_1^2 |z_1|^2 + \dots + c_m^2 |z_m|^2 + |z_{m+1}|^2 + \dots + |z_n|^2} \left( \sum_{i=1}^m c_i |z_i| \partial_{z_i} + \sum_{i=m+1}^n \bar{z}_i \partial_{z_i} \right).$$

Our goal will be to understand parallel transport around 0. As a first step, we examine the negative flow along the path  $\gamma_{vc} : \mathbb{R}_{>0} \rightarrow \mathbb{C}$  where  $\gamma_{vc}(t) = t$ . Let  $F_p$  be the fiber of our function over  $p$  and  $\phi_t : F_p \rightarrow F_{p-t}$  be the parallel transport map for  $p > t$ . Define  $T = \{z \in f_L^{-1}(\mathbb{R}_{>0}) : \lim_{t \rightarrow f_m(z)} \phi_t(z) = 0\}$  and  $L = \{z \in F_1 : \lim_{t \rightarrow 1} \phi_t(z) = 0\}$  to be the vanishing thimble and cycle respectively of  $f_L$ .

**Proposition 3.17.** *The vanishing thimble and cycle of  $f_L$  are*

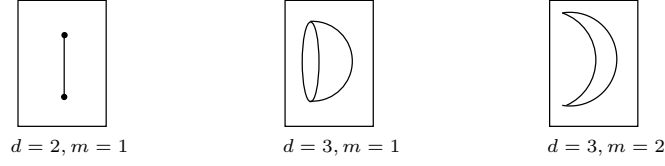
$$T = \mathbb{R}_{\geq 0}^m \times \mathbb{R}^{n-m},$$

$$L = \mathbb{R}_{\geq 0}^m \times \mathbb{R}^{n-m} \cap F_1 \approx \Delta_{m-1} \star S^{n-m-1},$$

where  $\Delta_{m-1}$  is the  $(m-1)$ -dimensional simplex,  $S^{n-m-1}$  is the sphere and  $\star$  is the join.

*Proof.* To find  $T$ , we obtain the flow lines of  $-\xi_m$  which gives the parallel transport along  $\gamma_{vc}$ . Let  $\nu_m$  be the vector field

$$(64) \quad \nu_m = -(c_1^2 |z_1|^2 + \dots + c_m^2 |z_m|^2 + |z_{m+1}|^2 + \dots + |z_n|^2) \xi_m = \sum_{i=1}^m c_i |z_i| \partial_{z_i} + \sum_{i=m+1}^n \bar{z}_i \partial_{z_i}.$$

FIGURE 3. The vanishing cycle as a join  $L \approx \Delta_{m-1} \star S^{n-m-1}$ .

Any flow line of  $\nu_m$  is a reparameterized flow line of  $-\xi_m$ . To see that elements with non-zero imaginary part do not lie in the thimble, we let  $g : \mathbb{C}^n \rightarrow \mathbb{R}$  be  $g(z) = \|Im(z)\|^2$  and suppose  $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}^n$  is a flow line of  $\nu_m$  with  $\alpha(0) = (\lambda_1, \dots, \lambda_n)$ . Note that  $\alpha'(0) = Re(\alpha'(0)) + iIm(\alpha'(0)) = [Re(\alpha)]'(0) + [Im(\alpha)]'(0)$  so that

$$(65) \quad [Im(\alpha)]'(0) = (0, \dots, 0, Im(\lambda_{m+1}), \dots, Im(\lambda_n))$$

and

$$(66) \quad (g \circ \alpha)'(0) = \sum_{i=m+1}^n Im(\lambda_i)^2 \geq 0.$$

This implies that flow lines have non-decreasing imaginary norm, so those that flow to zero must be in  $\mathbb{R}^n$ . But for  $(r_1, \dots, r_n) \in \mathbb{R}^n$ , there is an explicit solution to  $\nu_m$  given by

$$(67) \quad \alpha(t) = (c_1 r_1 e^{-\text{sign}(r_1)t}, \dots, c_m r_m e^{-\text{sign}(r_m)t}, r_{m+1} e^{-t}, \dots, r_n e^{-t}).$$

From this solution, it is clear that  $T = \mathbb{R}_{>0}^m \times \mathbb{R}^{n-m}$ .

The fact that  $\mathbb{R}_{>0}^m \times \mathbb{R}^{n-m} \cap F_1$  is the stated join follows from the general argument given in 4.11.  $\square$

We illustrate a few examples of these vanishing cycles in figure 3. In general, one would hope that these cycles could appear as natural objects in a Fukaya-Seidel category, perhaps with a partial wrapping around the stratifying divisors.

We now give a description of the monodromy map around the stratified Morse critical value. We will examine only the case where  $c_i = 1$  for every  $m+1 \leq i \leq n$ . The case of a more general  $L$  only affects the isotopy class of the map if a restrictive  $\partial$ -frame group is considered, otherwise, we may isotope to this case. Let  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be given by the map

$$(68) \quad \varphi(w_1, \dots, w_n) = \left( \frac{1}{2}w_1^2, \dots, \frac{1}{2}w_m^2, w_{m+1}, \dots, w_n \right).$$

Observe that  $\tilde{f}_L := f_L \circ \varphi(w_1, \dots, w_n) = 1/2 \sum w_i^2$  and that  $\varphi^* \omega_1 = \omega_{st}$  so that the diagram

$$(69) \quad \begin{array}{ccc} (\mathbb{C}^n, \omega_{st}) & \xrightarrow{\varphi} & (\mathbb{C}^n, \omega_1) \\ \tilde{f}_L \downarrow & & f_L \downarrow \\ \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \end{array}$$

commutes. This immediately implies that, off of  $D(m)$ , the parallel transport vector fields are mapped to each other via  $\varphi$ . In fact, it is easy to show that this description extends over  $D(m)$ .

The symplectic monodromy around 0 relative to the function  $\tilde{f}_L$  describes a spherical twist as described in [54]. Furthermore, this map is invariant with respect to the  $(\mathbb{Z}/2\mathbb{Z})^m$  action. In order to understand the monodromy for  $f_L$  on the quotient, we take a moment to recall the formulation of the spherical twist with respect to  $\tilde{f}_L$ . We take  $T^*S^{n-1} \approx \tilde{f}_L^{-1}(1)$  to be the cotangent bundle of the  $(n-1)$ -sphere. Taking  $g$  to be the constant curvature 1 metric on  $S^{n-1}$  we have the dual metric  $g^*$  on  $T^*S^{n-1}$  and consider  $H : T^*S^{n-1} \rightarrow \mathbb{R}$  to be the Hamiltonian generating geodesic flow [50] and  $\xi_H$  its vector field. By rescaling  $\xi_H$  off of the zero section  $Z \subset T^*S^{n-1}$ , one obtains a vector field whose time one flow can be smoothly extended to all of  $T^*S^{n-1}$  by taking the antipode map on  $Z$ . The resulting map  $\tilde{\phi}$  has support on a ball neighborhood of  $Z$  and is Hamiltonian isotopic to the spherical twist.

We would like to utilize this description to understand the stratified case above. Let  $\phi : f_L^{-1}(1) \rightarrow f_L^{-1}(1)$  be the monodromy map for  $f_L$  and let  $G = (\mathbb{Z}/2\mathbb{Z})^m$  be the group acting on  $T^*S^{n-1}$  by multiplying the  $i$ -th coordinate by  $\pm 1$ . We decompose  $Z$  into  $2^m$  regions  $Z = \cup_{g \in G} Z_g$  defined as

$$(70) \quad Z_0 = \{(w_1, \dots, w_n) \in S^{n-1} : w_i \geq 0 \text{ for all } 1 \leq i \leq m\},$$

and  $Z_g = g \cdot Z_0$ . We let  $T^*Z_0$  consist of pairs  $(w, v)$  such that, if  $w \in \partial Z_0$ , then  $v(\nu) \geq 0$  for all inward pointing tangent vectors  $\nu \in T_w Z_0$ . Then  $T^*Z_0$  forms a fundamental domain of  $f_L^{-1}(1)$  ramified over the boundary  $\partial Z_0$ .

Now, by restricting  $\tilde{\phi}$  to any fiber  $T_p S^{n-1}$  and projecting to  $Z$ , we obtain a decomposition of each such fiber  $T_p S^{n-1} = \cup_{g \in G} Z_{p,g}$ . Identifying  $T^*Z_0$  with  $f_L^{-1}(1)$ , the monodromy map  $\phi$  takes  $(p, v) \in Z_{p,g}$  to  $g^{-1}\varphi(p, v)$ . Qualitatively, we observe that a given fiber of  $T_p^*Z_0 \approx T_p^*L$  wraps around the zero section  $2^m$  times with one crossing. The map on the vanishing cycle is seen to be the join of the identity on the simplex with the antipode map on the sphere.

**3.4.  $\partial$ -framed Lefschetz pencils.** In this section we address certain transitions in framings for symplectomorphisms arising in monodromy calculations. We assume  $(\mathcal{Y}, D)$  is a Kähler orbifold with standard normal crossing divisor  $D = D_1 + \dots + D_k$ , i.e. it is a symplectic orbifold with a specified  $J \in \mathcal{J}$  that is integrable everywhere.

Take  $\mathcal{C}$  to be a 1-dimensional DM stack with coarse space  $\mathbb{P}^1$ . Let  $\pi : \mathcal{Y} \rightarrow \mathcal{C}$  be a map with determinant values  $\text{Det}(\pi)$ . Recall that these are values for which  $\pi$  is either singular, or where  $\pi|_{\cap_{i \in I} D_i}$  is singular.

**Definition 3.18.** We will say that  $\pi$  defines a  $\partial$ -framed Lefschetz pencil if  $\omega \in \Omega^2(\mathcal{Y})$  is isotopic to some  $\tilde{\omega}$  for which  $D$  is horizontal and such that there is a covering  $\{U_i\}$  of  $\mathcal{C}$  such that  $\pi : \pi^{-1}(U_i) \rightarrow U_i$  is either a smooth proper fibration, a normal crossing degeneration or a stratified Morse function for every  $i$ . If  $(\mathcal{Y}, D)$  is a standard Kähler stack with resolving collection  $\mathcal{R}$ , we say that  $\pi$  is a  $\partial$ -framed Lefschetz pencil if  $\pi \circ \psi_\varepsilon : \tilde{\mathcal{Y}} \rightarrow \mathcal{C}$  is a  $\partial$ -framed Lefschetz pencil for every  $(\tilde{\mathcal{Y}}, \tilde{D}) \in \mathcal{R}$ .

We note that the definition of Lefschetz pencil given in [24] is generalized by the definition above. The notion of a partial Lefschetz fibration given in [40] can also be put in this framework. However, our principal example of a framed pencil is obtained from considering stacky curves in  $\mathcal{X}_{\Sigma(A)}$  where  $A$  satisfies some basic conditions.

**Theorem 3.19.** Suppose  $A \subset \mathbb{Z}^d$  defines the marked polytope  $(Q, A)$  such that for every face  $F$  of  $Q$  either  $\text{orb}_F$  has a smooth neighborhood, or  $(F, A \cap F)$  is dual

*defect.* Let  $(\mathcal{Y}_A, \partial\mathcal{Y}_A) \subset \mathcal{X}_{\Theta(A)}$  be the universal toric hypersurface with boundary. Suppose  $\mathcal{C}$  is 1-dimensional and  $i : \mathcal{C} \rightarrow \mathcal{X}_{\Sigma(A)}$  is an embedding which transversely intersects the  $E_A^s$  determinant. Then  $\pi : i^*(\mathcal{Y}_A, \partial\mathcal{Y}_A) \rightarrow \mathcal{C}$  is a  $\partial$ -framed pencil.

*Proof.* Let  $E_A = \prod_{A_i \subset Q} \Delta_{A_i}$  be the product decomposition of  $E_A$  as in [46]. Recall that this is indexed by faces  $Q_i = \text{Conv}(A_i) \subset Q$ . Under the conditions above, if the orbit  $X_{A_i}$  does not admit a smooth neighborhood then  $\Delta_{A_i}$  is constant. So we may assume that all  $A_i$  indexing  $\Delta_{A_i}$  do admit smooth neighborhoods.

Since every intersection point  $p \in i(\mathcal{C}) \cap \{\Delta_{A_i} = 0\}$  is transverse, we have that the point  $p$  is a smooth point of  $\Delta_{A_i}$  and not in  $\{\Delta_{A_i} = 0\}$  for any other  $i$ . This implies, by proposition 3.15, and the definition of discriminant, that  $\pi : i^*(\mathcal{Y}_A, \partial\mathcal{Y}_A) \rightarrow U$  is a stratified Morse singularity in a neighborhood  $U$  of  $p$ .

For every  $p \in i(\mathcal{C}) \cap \partial\mathcal{X}_{\Sigma(A)}$ , theorems 2.18 and 2.19 imply that there is a neighborhood  $U$  of  $p$  such that  $\pi : i^*(\mathcal{Y}_A, \partial\mathcal{Y}_A) \rightarrow U$  is a toric hypersurface degeneration of  $\mathcal{Z}_A(q) = \pi^{-1}(q)$  for  $q \in U - \{p\}$ .  $\square$

All of the results on symplectomorphisms will be obtained by parallel transport in a  $\partial$ -framed Lefschetz pencil. However, the parallel transport map occurs naturally as a functor in higher dimensional settings. We take a moment to fix notation for the general set up, and quickly return to the 1-dimensional case afterwards.

Given a stack  $\mathcal{X}$  with atlas  $(U_\beta, G_\beta, \pi_\beta)_{\beta \in \mathcal{B}}$ , let  $\Pi(\mathcal{X})$  be the path category of  $\mathcal{X}$  defined by taking objects  $p \in \cup U_\beta$  to be objects and morphisms  $\text{Hom}(p, q) = \{\gamma : [0, 1] \rightarrow \mathcal{X} : \gamma(0) = p, \gamma(1) = q\}$ . We can think of this category as an  $(\infty, 1)$ -category, as morphisms do not compose associatively.

Given a bundle  $\pi : (\mathcal{Y}, \partial\mathcal{Y}) \rightarrow \mathcal{X}$  of standard symplectic stacks over  $\mathcal{X}$  and a symplectic connection which preserves their boundaries, we write parallel transport as a functor

$$(71) \quad \mathbf{P} : \Pi(\mathcal{X}) \rightarrow \mathbf{Symp}$$

where  $\mathbf{Symp}$  is the category of standard symplectic stacks. This map takes  $p \in \mathcal{X}$  to  $\pi^{-1}(p)$  and a morphism to the map obtained by parallel transport. We will abuse notation and also write  $\mathbf{P} : \Omega_p(\mathcal{X}) \rightarrow \text{Symp}(\pi^{-1}(p), \partial\pi^{-1}(p))$  for the restriction to based loops. As indicated by theorem 3.19, the primary example we consider is  $\mathcal{X} = (\mathcal{X}_{\Sigma(A)} - \mathcal{E}_A)$ . Using this theorem and this general parallel transport map, we define:

**Definition 3.20.** For any point  $p \in (\mathcal{X}_{\Sigma(A)} - \mathcal{E}_A)$  let  $\mathbf{G}_p \subset \pi_0(\text{Symp}(\mathcal{Z}_A(p), \partial\mathcal{Z}_A(p)))$  be the group of components of the image  $\mathbf{P}(\Omega_p(\mathcal{X}_{\Sigma(A)} - \mathcal{E}_A))$ .

For any  $\partial$ -framed Lefschetz pencil, for  $q \in \mathcal{C} - \text{Det}(\pi)$  and take  $\mathcal{Z}_q = \pi^{-1}(q)$  to be a smooth fiber with  $\partial\mathcal{Z}_q = \mathcal{Z}_q \cap D$ . If the  $q$  is a chosen base point, we simply write  $\mathcal{Z}$  and  $\partial\mathcal{Z}$ . Note that the definition above ensures that every fiber outside  $\text{Det}(\pi)$  transversely intersects  $D$ , so  $(\mathcal{Z}, \partial\mathcal{Z})$  is a symplectic orbifold with standard normal crossing divisor.

The connection given by the modified symplectic form  $\tilde{\omega}$  yields a parallel transport map that preserves the boundary, which we write as

$$(72) \quad \mathbf{P} : \Omega_q(\mathcal{C} - \text{Det}(\pi)) \rightarrow \text{Symp}(\mathcal{Z}, \partial\mathcal{Z}),$$

where  $\Omega_q$  is notation for piecewise smooth based loops at  $q$ .



A key point is that when we examine local monodromy, we may utilize the local model descriptions to analyze the symplectomorphisms as framed maps with respect to a reasonable  $\partial$ -frame group. However, as we extend to the global pencil, these maps lose their framing in the holonomy. Another way of saying this is that, up to homotopy, if we omit a point  $q_\infty \in \mathcal{C} - \text{Det}(\pi)$ , we may define a  $\partial$ -frame group  $\mathbf{F}$  which is tightly controlled on parts of  $\partial\mathcal{Z}$  and obtain, up to homotopy, a lift

$$(73) \quad \mathbf{P}_\infty : \Omega_q(\mathcal{C} - \text{Det}(\pi) - \{q_\infty\}) \rightarrow \text{Symp}^{\mathbf{F}}(\mathcal{Z}, \partial\mathcal{Z}).$$

However, to extend the map to  $\Omega_q(\mathcal{C} - \text{Det}(\pi))$ , we need to relax our  $\partial$ -frame group  $\mathbf{F}^{\text{rel}}$ .

Let us now define the  $\partial$ -frame group of a  $\partial$ -framed Lefschetz fibration. For this, we let  $\partial\mathcal{Z} = \sum_{i=1}^k \tilde{D}_i$ .

**Definition 3.21.** A boundary component  $D_i$  is called rigid if there exists a symplectic orbifold  $\mathcal{C}_i$  and a branched cover  $\iota : \mathcal{C}_i \rightarrow \mathcal{C}$  such that  $D_i - \partial D_i \approx (\tilde{D}_i - \partial \tilde{D}_i) \times \mathcal{C}_i$  and  $\iota^*(\pi|_{D_i - \partial D_i})$  is projection to the second factor.

We say that a face  $F \subset Q$  is a simplicial face if  $F$  is a face of  $Q$  and  $F \cap A$  is an affinely independent set. The following proposition may be seen from the fact that simplicial faces of  $\mathcal{X}_Q$  occur in trivial families as substacks of  $\pi : \mathcal{Y}_A \rightarrow \mathcal{X}_{\Sigma(A)}$ .

**Proposition 3.22.** *If  $i : \mathcal{C} \rightarrow \mathcal{X}_{\Sigma(A)}$  pulls back  $\pi : \mathcal{Y}_A \rightarrow \mathcal{X}_{\Sigma(A)}$  to a  $\partial$ -framed Lefschetz fibration and  $D_i$  is a divisor associated to a simplicial facet of  $(Q, A)$ , then  $D_i$  is rigid.*

Now, let  $\text{Det}(\pi) = \{q_1, \dots, q_N\}$  and take  $B_\varepsilon(p)$  to be a disc of radius  $\varepsilon$  around  $p$ . We take  $\mathcal{B} = \{\gamma_1, \dots, \gamma_N\}$  to be a set of embedded paths from  $[0, 1]$  to  $\mathcal{C}$  such that  $\gamma_i(0) = q$ ,  $\gamma_i(1) = q_i$  and, for  $i \neq j$ ,  $\gamma_i(t) = \gamma_j(s)$  iff  $t = s = 0$ . We also assume that  $\gamma'_i(0)$  is ordered counter-clockwise. Such a collection is known as a distinguished basis of paths [15]. For any such distinguished basis and any  $\gamma_i$ , we define the loop  $\gamma_i^\varepsilon$  by following  $\gamma_i$  until reaching a distance of  $\varepsilon$ , circling around the boundary of  $B_\varepsilon(q_i)$  counter-clockwise and following  $\gamma_i$  back to  $q$ . For  $\varepsilon$  sufficiently small, we may apply a Hamiltonian perturbation to the connection so that  $\phi_i = \mathbf{P}_\infty(\gamma_i^\varepsilon)$  is a degeneration monodromy map or stratified Morse monodromy map as presented in the previous sections. We divide  $\{1, \dots, N\} = I_d \cup I_m$  into those points of toric degeneration monodromy and stratified Morse values respectively.

For  $i \in I_m$ , we let  $L_i \subset (\mathcal{Z}, \partial\mathcal{Z})$  be the vanishing cycle pulled back along  $\gamma_i$  and write  $S_i = \{j : L_i \cap \tilde{D}_j \neq \emptyset\}$  for the set of divisors that intersect the vanishing cycle of  $\gamma_i$ . Let  $K_i$  be a relatively compact neighborhood of  $\partial L_i$  and  $K = \cup_i K_i$ . By the results of section 3.3, we have that  $\phi_i$  can be viewed as a symplectomorphism with support in  $K_i$ . For  $i \in I_d$ , let  $\eta_i : Q \rightarrow \mathbb{R}$  be the defining toric degeneration at  $q_i$  and take  $S_i = \{j : \eta_i \text{ is not linear on } \partial Q_j\}$ . In other words, the degeneration of  $\mathcal{Z}$  at  $q_i$  also degenerates  $\tilde{D}_j$ . We take  $T = \{1, \dots, k\} - \cup_i S_i$  and observe that every boundary divisor  $\tilde{D}_j$  is rigid if  $j \in T$ , as in the previous proposition.

Now, for  $i \in I_d$ , we have that  $\eta_i \in (\mathbb{Z}^A)^\vee$  defines affine functions  $T_{i,j}$  on the components  $Q_j$  of the subdivision defined by  $\eta_i$ . For each  $k \in S_i$ , let  $\nu_k$  be the normal direction to  $\partial Q_k$  and define  $d_{i,k}$  to be  $\partial_{\nu_k} T_{i,k}$ . Let

$$(74) \quad \mathbb{R}_{\eta_i}^k = \{(r_1, \dots, r_k) : r_j \in \mathbb{R} \text{ for } j \notin S_i, r_j \equiv 0 \pmod{d_{i,j}}\},$$

$\mathbb{R}_\pi^k = \times_{i \in I_d} \mathbb{R}_{\eta_i}^k$ , and  $\mathbf{T}_{\eta_i} = \{\tau(\mathbf{r}) : \mathbf{r} \in \mathbb{R}_{\eta_i}^k\}$ . We take  $\mathbf{T}_\pi$  to be the group generated by the subgroups  $\mathbf{T}_{\eta_i}$  over all  $i \in I_d$ .

**Definition 3.23.** Let  $\pi : \mathcal{Y} \rightarrow \mathcal{C}$  be a  $\partial$ -framed Lefschetz pencil. The  $\partial$ -frame group  $\mathbf{F} \in \text{Symp}(N_{\partial\mathcal{Z}}\mathcal{Z}/\partial\mathcal{Z})$  associated to  $\pi$  is given by the collection of maps whose restrictions to  $\cup_{j \in T} \tilde{D}_j$  are contained in the restriction of  $\mathbf{T}_\pi$ .

We note that if every components of the boundary of  $Q$  is degenerated or bounds a stratified Morse singularity over some  $q_i \in \text{Det}(\pi)$ , then  $\mathbf{F}$  equals  $\text{Symp}(N_{\partial\mathcal{Z}}\mathcal{Z}/\partial\mathcal{Z})$ . On the other hand, if  $Q$  is simplicial, then  $\mathbf{F}$  is a discrete subgroup of  $\mathbf{T}$ . Ideally, one would like to obtain more control over the  $\partial$ -framing for the non-rigid boundary components and incorporate this into a formula such as the one in 3.25, but this is currently not within our sight. However, we may use the results of the previous sections to prove:

**Proposition 3.24.** *If  $\pi : \mathcal{Y} \rightarrow D$  is a  $\partial$ -framed Lefschetz pencil and  $q_\infty$  is chosen as above, then there exists a symplectic connection for which*

$$(75) \quad \mathbf{P}_\infty : \Omega_q(\mathcal{C} - \text{Det}(\pi) - \{q_\infty\}) \rightarrow \text{Symp}^{\mathbf{F}}(\mathcal{Z}, \partial\mathcal{Z}),$$

where  $\mathbf{F}$  is the  $\partial$ -frame group associated to  $\pi$ .

*Proof.* For every  $i \in I_m$ , we have, by definition, that the divisors supporting the degenerate point are horizontal with respect to  $\tilde{\omega}$ . As described in section 3.3, the monodromy is Hamiltonian isotopic to a map supported on the relatively compact neighborhood  $K_i$ . Since this neighborhood can be made to be disjoint from all  $\tilde{D}_i$  with  $i \in T$ , the restriction of the map to the framing group  $\mathbf{F}$  is well defined. Indeed, the monodromy map is the identity on any rigid  $\tilde{D}_i$ .  $\square$

We observe that for  $\partial$ -frame groups associated to  $\partial$ -framed Lefschetz pencils, the exact sequence in proposition 3.7 yields the fiber sequence

$$(76) \quad \text{Symp}^{\mathbf{F}}(\mathcal{Z}, \partial\mathcal{Z}) \rightarrow \text{Symp}^{\mathbf{F}^{rel}}(\mathcal{Z}, \partial\mathcal{Z}) \rightarrow \mathbb{R}^k / \mathbb{R}_\pi^k.$$

Note that the last group is homotopic to  $(S^1)^t$  where  $t \leq |I_d|$ .

Now, write  $\gamma$  for the concatenation  $\gamma_N \circ \dots \circ \gamma_1$  which is independent of the distinguished basis. Let  $N(\gamma)$  be the normalizer of  $\gamma$  in the group  $\pi_1(\mathcal{C} - \text{Det}(\pi) - \{q_\infty\})$ . We write  $F_k$  for the free group on  $k$  letters and, utilizing proposition 3.5, we obtain the commutative diagram

$$\begin{array}{ccccccc} N(\gamma) & \longrightarrow & F_N & \longrightarrow & F_{N-1} & \longrightarrow & 1 \\ \varrho \downarrow & & [\mathbf{P}_\infty] \downarrow & & [\mathbf{P}] \downarrow & & \\ \pi_1((S^1)^t) & \xrightarrow{\delta} & \pi_0(\text{Symp}^{\mathbf{F}}(\mathcal{Z}, \partial\mathcal{Z})) & \longrightarrow & \pi_0(\text{Symp}^{\mathbf{F}^{rel}}(\mathcal{Z}, \partial\mathcal{Z})) & \longrightarrow & 1 \end{array}.$$

The image of  $\gamma$  under  $\varrho$  will be of particular importance in the coming sections. Under the restrictions laid out above though, there is an explicit formula for this map:

**Proposition 3.25.** *If  $\pi : \mathcal{Y} \rightarrow \mathcal{C}$  is a  $\partial$ -framed Lefschetz pencil, then, for every  $i \in T$ , there exists a section  $s_i : \mathcal{C}_i \rightarrow D_i$  such that*

$$(77) \quad \varrho(\gamma) = \sum_{i=1}^k a_i e_i$$

where  $a_i = \int_{\mathcal{C}_i} s_i^*(c_1(N_{\mathcal{Y}}\tilde{D}_i))$  and  $e_i$  is the loop in  $\text{Symp}(N_{\partial\mathcal{Z}}\mathcal{Z}/\partial\mathcal{Z})$  corresponding to rotation around  $\tilde{D}_i$ .

*Proof.* By definition of rigid boundary divisor, for every  $i \in T$ , the restriction of  $\pi$  to  $D_i$  is trivial onto its image  $\mathcal{C}_i \subset \mathcal{C}$ . Over the contractible subset  $U_0 = \mathcal{C}_i - \{q_\infty\}$ , we may trivialize  $N_{\tilde{D}_i} \mathcal{Z}_{q'}$  uniformly for all  $q' \in \mathcal{C} - \{q_\infty\}$ . Likewise, in an open neighborhood  $U_1$  of  $q_\infty$ , we may trivialize  $N_{\tilde{D}_i} \mathcal{Z}_{q'}$ . Taking a circle  $\delta$  in the intersection  $U_0 \cap U_1$ , we see the transition function may be made uniform fiberwise. The multiplicative factor of the transition function on the normal bundle restricted to  $\delta$  is given by the transition function on  $N_{D_i} \mathcal{Y}$  restricted to  $\mathcal{C}_i \times \{p\} \subset D_i$ . The winding number is given by the Chern number of  $s_i^*(N_{D_i} \mathcal{Y})$  where  $j : \mathcal{C}_i \rightarrow D_i$  is a section. On a normal neighborhood of  $\tilde{D}_i$  in  $\mathcal{Z}$ , this is the restriction of  $\tau(\mathbf{t})$  to  $\tilde{D}_i$  where  $\mathbf{t} = (0, \dots, 0, a_i, 0, \dots, 0)$ . Adding these together for each rigid component yields the claim.  $\square$

We end this section by defining a subgroup of the framed symplectomorphism group of a toric hypersurface.

**Definition 3.26.** Let  $A \subset \mathbb{Z}^d$  satisfy the hypothesis of 3.19 and  $i : \mathcal{C} \rightarrow \mathcal{X}_{\Sigma(A)}$  an embedded curve. The group  $\mathbf{G}_{\mathcal{C}} = \mathbf{P}_\infty(i_*(\Omega_p(\mathcal{C} - \mathcal{C} \cap E_A^s))) \subset \mathcal{Z}_A(i(p))$  will be called the  $\mathcal{C}$  subgroup of  $\text{Symp}(\mathcal{Z}_A(i(p)), \partial \mathcal{Z}_A(i(p)))$ .

One of our stated goals is to understand generators and relations for the group  $\mathcal{G}_A := \mathbf{P}(\Omega_p(\mathcal{X}_{\Sigma(A)} - E_A^s))$ . We may reduce the complexity of this problem by examining  $\partial$ -framed Lefschetz pencils and their monodromy.

**Proposition 3.27.** *Assume that  $\mathcal{X}_{\Sigma(A)}$  does not have generic isotropy. For any embedded  $i : \mathcal{C} \rightarrow \mathcal{X}_{\Sigma(A)}$  for which  $[\mathcal{C}]$  is dual to a very ample line bundle  $\mathcal{L}$ , the group  $\pi_0(\mathbf{G}_{\mathcal{C}})$  surjects onto  $\mathbf{G}$ .*

*Proof.* For a very ample line bundle  $\mathcal{L}$  with equivariant linear system  $V$  we have an embedding on the coarse space  $j : X_{\Sigma(A)} \rightarrow \mathbb{P}(V)$ . The Lefschetz hyperplane theorem gives a surjection from the fundamental group of the curve arising from a linear section of  $j(X_{\Sigma(A)} - E_A)$  and that of  $\mathcal{X}_Q - E_A$ . But if  $\mathcal{B}$  denotes the points with non-trivial isotropy on  $\mathcal{X}_{\Sigma(A)}$  and  $B$  its coarse space, then  $\pi_1(X_{\Sigma(A)} - E_A - B)$  surjects onto  $\pi_1(X_{\Sigma(A)} - E_A)$  yielding the result.  $\square$

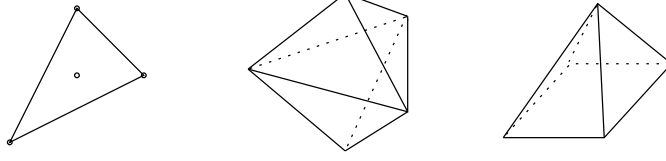
#### 4. THE CIRCUIT RELATION

This section will give one main result of this paper which is a detailed description of a class of relations that occur in the symplectic mapping class groups of toric hypersurfaces. These relations involve a combination of toric degeneration monodromy and twisting about a stratified Morse singularity. Near the toric degeneration points of our  $\partial$ -framed Lefschetz pencil, we give a partial description of the vanishing cycle of the stratified Morse singularity, giving further geometric content to the relation. Finally, we explain how to connect various relations into a finite presentation.

**4.1. Circuit stacks.** We begin by recalling the definition and basic properties of a circuit from [46] and detailing the Lafforgue and secondary stack of a circuit. The map  $\pi : \mathcal{X}_{\Theta(A)} \rightarrow \mathcal{X}_{\Sigma(A)}$  will be also be reexpressed in concrete terms and its monodromy will be studied.

For what follows, we will assume that  $\Lambda \approx \mathbb{Z}^d$  is a rank  $d$  affine lattice.

**Definition 4.1.** A circuit  $A \subset \Lambda$  is a finite subset for which every proper subset is affinely independent.



$$\begin{array}{lll} \sigma_A = (1, 3) & \sigma_A = (2, 3) & \sigma_A = (2, 2; 1) \\ \mathbf{a} = (3, -1, -1, -1) & \mathbf{a} = (3, 3, -2, -2, -2) & \mathbf{a} = (-1, -1, 1, 1, 0) \end{array}$$

We will say that a subset  $A \subset \Lambda$  has rank  $r$  if  $|A| - 2 = \text{rk}(\text{Aff}_{\mathbb{Z}}(A)) = r$  where  $\text{Aff}_{\mathbb{Z}}(A)$  is the integral affine span of  $A$ . A circuit is non-degenerate if its rank and that of  $\Lambda$  coincide. We will consider both non-degenerate and degenerate circuits, but they are related by the following definition.

**Definition 4.2.** An extended circuit is a subset  $A \subset \Lambda$  such that  $|A| = d + 2$  and  $\text{rk}(\text{Aff}(A)) = d$ . The core of an extended circuit  $A$  is the unique circuit  $A - A_0$  contained in  $A$ .

Alternatively, an extended circuit is an affinely spanning subset  $A = \{\alpha_0, \dots, \alpha_{d+1}\}$  whose lattice of affine relations has rank 1, generated by  $\mathbf{a} = (a_0, \dots, a_{d+1})j \in \mathbb{Z}^{d+2}$  where

$$(78) \quad \sum_{i=0}^{d+1} a_i \alpha_i = 0,$$

$$(79) \quad \sum_{i=0}^{d+1} a_i = 0.$$

Our convention is that the greatest common divisor of the  $a_i$  is  $|K_{A^e}|$ . We note that this implies the volume of  $Q$  is then

$$(80) \quad v_A := \text{Vol}(Q) = \pm \sum_{\alpha_i \in A_{\pm}} a_i.$$

where we normalize the volume of the unit simplex to 1.

Given this relation, we may write  $A$  as the disjoint union  $A = A_- \cup A_0 \cup A_+$  where  $\alpha_i \in A_{\pm}$  iff  $\pm a_i > 0$  and  $\alpha_i \in A_0$  iff  $a_i = 0$ . The signature of an extended circuit is defined to be  $\sigma(A) = (|A_+|, |A_-|; |A_0|)$ . When  $A$  is a circuit, it is clear that  $|A_0| = 0$  and we then write  $\sigma(A) = (|A_+|, |A_-|)$ . Note that the signature depends on the orientation of  $\mathbf{a}$  up to transposing  $|A_+|$  and  $|A_-|$ .

To any extended circuit  $A$ , there are precisely two regular triangulations  $T_{\pm}$  of  $(Q, A)$  given by

$$(81) \quad T_{\pm} = \{\text{Conv}(A - \{\alpha_i\})\}_{i \in A_{\pm}}.$$

These triangulations are marked by  $A$  unless  $|A_+| = 1$  or  $|A_-| = 1$ , in which case the respective triangulation is marked by  $A - A_{\pm}$ .

While the next two sections will deal with the geometry of extended circuits  $(Q, A)$  in isolation, the primary reason for us to investigate them is how they relate to a larger marked polytope  $(\mathbf{Q}, \mathbf{A})$  containing  $(Q, A)$ . The key fact in this regard is that every edge of the secondary polytope  $\Sigma(\mathbf{A})$  corresponds to a circuit modification. We recall this theorem and the necessary definitions from [46].

**Definition 4.3.** Let  $\mathbf{T}$  be a triangulation of  $(\mathbf{Q}, \mathbf{A})$  and  $A \subset \mathbf{A}$  a circuit with triangulations  $T_{\pm}$ . We say that  $\mathbf{T}$  is positively (resp. negatively) supported on  $A$  if the following conditions hold:

- (i)  $T_+$  (resp.  $T_-$ ) consists of faces of simplices in  $\mathbf{T}$ .
- (ii) For every  $J \subset \mathbf{A}$ , if  $\sigma \in T_+$  (resp.  $T_-$ ) with  $J \cup \sigma$  a maximal simplex of  $\mathbf{T}$  then  $J \cup \sigma' \in \mathbf{T}$  for every  $\sigma' \in T_+$  (resp.  $T_-$ ).

For any such  $J$  above, we say that  $J \cup A$  is a separating extended circuit of  $\mathbf{T}$ .

If  $\mathbf{T}$  is positively supported on  $A$ , then one may define a new triangulation  $s_A(\mathbf{T}) := \mathbf{T}'$  which is negatively supported on  $A$  by changing the triangulations of every separating extended circuit. Such a change is referred to as a circuit modification along  $A$ .

**Theorem 4.4** ([46], 7.2.10). *Let  $\mathbf{T}$  and  $\mathbf{T}'$  be two regular triangulations of  $(\mathbf{Q}, \mathbf{A})$ . The vertices  $\phi_{\mathbf{T}}, \phi_{\mathbf{T}'} \in \Sigma(\mathbf{A})$  are joined by an edge if and only if there is a circuit  $A \subset \mathbf{A}$  such that  $\mathbf{T}$  is supported on  $A$  and  $\mathbf{T}' = s_A(\mathbf{T})$ .*

This theorem indicates that if we aim to understand the symplectic monodromy of a hypersurface as we loop around  $E_{\mathbf{A}}$ , it is a reasonable first step to understand the monodromy around circuits, extended circuits and, more generally, circuit modifications.

We now take a moment to study basic properties of the toric stack  $\mathcal{X}_Q$  associated to an extended circuit by investigating the normal cone to  $Q$ . Suppose  $\Gamma = \Gamma_1 \oplus \Gamma_2$  where the rank of  $\Gamma_i$  is  $d_i$  and  $(Q_i, A_i)$  are marked polytopes. If  $(Q_2, A_2)$  is a simplex, we say that  $(Q_1 + Q_2, A_1 \cup A_2)$  is a  $d_2$ -simplicial extension of  $(Q_1, A_1)$ . By this definition, an extended circuit  $A$  of signature  $(p, q; r)$  is an  $r$ -simplicial extension of its core. If both  $(Q_i, A_i)$  are simplices, we say  $(Q_1 \oplus Q_2, A_1 \oplus A_2)$  is a  $(d_1, d_2)$ -prism.

**Proposition 4.5.** *If  $A \subset \Lambda$  is an extended circuit with signature  $(p, q; r)$ , then the polar polytope to  $Q$  is an  $r$ -simplicial extension of a  $(p, q)$ -prism.*

*Proof.* We begin with the case of a non-degenerate circuit  $A$ . We claim that, in this case, any facet of  $Q$  arises as the convex hull  $F_{ij} = \text{Conv}(A - \{\alpha_i, \alpha_j\})$  where  $\alpha_i \in A_+$  and  $\alpha_j \in A_-$ . To see this, first observe that every such  $F_{ij}$  is a facet which is clear from the description of the triangulations  $T_{\pm}$  above. observe that the element

$$(82) \quad u_A := \frac{1}{v_A} \sum_{\alpha_i \in A_+} a_i \alpha_i = \frac{1}{v_A} \sum_{\alpha_j \in A_-} a_j \alpha_j$$

lies in the convex hull of both  $A_+$  and  $A_-$  and the interior of  $A$ . Thus no facet  $F$  can contain  $A_+$  or  $A_-$  which implies that there exists an  $i$  and  $j$  with  $F_{ij} \subset F$  implying  $F_{ij} = F$ .

Let  $\tilde{A}_{\pm} = \{\alpha - u_A : \alpha \in A_{\pm}\}$  and  $\Lambda_{\pm} = \text{Lin}_{\mathbb{R}}(\tilde{A}_{\pm})$ . It is obvious that  $\tilde{A}_{\pm}$  are both simplices and we write  $B_{\pm} = \{v : v(w) \geq -1 \text{ for } w \in \tilde{A}_{\pm}\} \subset \Lambda_{\pm}^{\vee} \otimes \mathbb{R}$  for their polar sets. Then a basic argument gives  $\Lambda \otimes \mathbb{R} = \Lambda_+ \oplus \Lambda_-$  and that the polar dual of  $A$  is  $B_+ \oplus B_-$ .

Now, if  $A$  has signature  $(p, q; r)$ , we take  $\tilde{A}_0 = \{\alpha - u_A : \alpha \in A_0\}$  and  $\Lambda_0 = \text{Lin}_{\mathbb{R}}\{\alpha - u_A : \alpha \in A_0\}$ . Since  $\tilde{A}_0$  is not full dimensional, there is no polar polytope in  $\Lambda_0^{\vee}$ , but we still have that  $\Lambda \otimes \mathbb{R} = \Lambda_+ \oplus \Lambda_- \oplus \Lambda_0$  and we have that  $\tilde{A}_0$  is a basis

for  $\Lambda_0$ . If  $B_0 \subset \Lambda_0^\vee$  denotes the negatives of the linear duals to  $\tilde{A}_0$ , then we have that the polar dual for  $A$  is the simplicial extension  $(B_+ \oplus B_-) + B_0$ .  $\square$

For later reference, we utilize the previous proposition to index the boundary facets of  $Q$ .

**Corollary 4.6.** *If  $(Q, A)$  is an extended circuit of signature  $(p, q; r)$ , then it has  $pq + r$  facets  $Q = \{b_{ij} : \alpha_i \in A_-, \alpha_j \in A_+\} \cup \{b_k : \alpha_k \in A_0\}$ .*

One important consequence of the proposition is that  $\mathcal{X}_Q$  fails to be smooth as a stack unless the signature of  $A$  has  $p = 1$ ,  $q = 1$  or is  $(2, 2; r)$ . Indeed, for a circuit  $(Q, A)$ , the maximal normal cones to  $Q$  are cones over products of simplices, and are therefore not simplicial. Nevertheless, as  $\mathcal{X}_Q$  is toric, it has a simplicial refinement which implies that  $(\mathcal{X}_Q, \partial\mathcal{X}_Q)$  is a standard symplectic stack.

We now examine the secondary and Lafforgue stacks associated to  $A$ . Applying equation 27, the triangulations  $T_\pm$  correspond to the vertices  $\phi_\pm \in \Sigma(A)$  given by

$$(83) \quad \phi_\pm = v_A \sum_{i=0}^{d+1} e_i \mp \sum_{\alpha_i \in A_\pm} a_i e_i.$$

So that, by equation 26,  $\Sigma(A) = \text{Conv}(\{\phi_-, \phi_+\})$  and

$$(84) \quad \Theta_p(A) = \text{Conv}(\{\phi_\pm + e_i : 0 \leq i \leq d+1\} \cup \{\phi_\pm\}).$$

To obtain  $\mathcal{X}_{\Theta(A)}$ , we take  $\Theta_p(A)$  and study the dual fan. We now give a geometric description of  $\mathcal{X}_{\Theta(A)}$  for an extended circuit  $A$ .

**Proposition 4.7.** *Given an extended circuit  $A \subset \Lambda$  of signature  $(p, q; r)$ , there is a group  $G_A$  such that  $\mathcal{X}_{\Theta(A)}$  is a stacky blow-up of  $\mathbb{P}^{d+1}/G_A$  along  $pq$  codimension 2 projective subspaces. If  $A$  is a circuit,  $G_A = 1$ .*

*The universal line bundle  $\mathcal{O}_A(1)$  and section are the pullbacks of  $\mathcal{O}(1)$  and  $s_A = \sum_{i=0}^{d+1} Z_i$ .*

*Proof.* We recall that  $\Theta_p(A) \subset \mathbb{R}^A$  is a polyhedron of dimension  $|A|$ . By proposition 2.13, the facets  $\overline{\Theta_p(A)}$  of  $\Theta_p(A)$  are in one to one correspondence with  $\{\varrho_A\} \cup \overline{\Theta_p(A)}^v \cup \overline{\Theta_p(A)}^h$  where  $\varrho_A = \sum e_\alpha^\vee$ , the elements of  $\overline{\Theta_p(A)}^v$  correspond to vertical hyperplanes and those of  $\overline{\Theta_p(A)}^h$  correspond to horizontal. The latter are indexed by a pointed coarse subdivision, but since  $A$  is an extended circuit, these are just triangulations  $T_\pm$  along with an element  $\alpha \in A_\pm$ . It is then simple to see that  $\{e_\alpha^\vee\}_{\alpha \in A_+ \cup A_-} = \overline{\Theta_p(A)}^h$ .

Recall that  $\overline{\Theta_p(A)}^v = \beta^\vee(\overline{Q}) = \{\beta^\vee(b_{ij}, n_{b_{ij}}) : \alpha_i \in A_-, \alpha_j \in A_+\} \cup \{\beta^\vee(b_k, n_{b_k}) : \alpha_k \in A_0\}$ . For a marked face  $(F, B)$  associated to  $b$ , we take  $r_b := [\text{Aff}_\mathbb{Z}(B) : \Lambda]$  for the lattice index. For any  $\alpha_k \in A_0$ , we have that  $F_k = A - \{\alpha_k\}$  is a face of  $Q$  implying

$$(85) \quad \bar{b}_k := \beta^\vee(b_k, n_{b_k}) = (v_A/r_{b_k})e_{\alpha_k}^\vee.$$

This follows from the fact that  $b_k(\alpha_k - F_k)r_{b_k} = v_A$ . On the other hand, if  $F_{ij}$  is the face associated to  $b_{ij}$ , we have that  $b_{ij}(\alpha_i - F_{ij})r_{b_{ij}} = \text{Vol}(\text{Conv}(A - \{\alpha_j\})) = a_j$  and  $b_{ij}(\alpha_j - F_{ij})r_{b_{ij}} = \text{Vol}(\text{Conv}(A - \{\alpha_i\})) = a_i$ . This implies that

$$(86) \quad \bar{b}_{ij} := \beta^\vee(b_{ij}, n_{b_{ij}}) = \frac{1}{r_{b_{ij}}} (a_j e_{\alpha_i}^\vee + a_i e_{\alpha_j}^\vee).$$

We now describe the fan  $\mathcal{F}_{\Theta_p(A)}$ , or equivalently, the abstract simplicial complex on  $\overline{\Theta_p(A)}$ . Note that the maximal cones are  $\{\sigma_{P_{\pm,i}}\}$  where  $\sigma_{P_{\pm,i}}$  consists of all pointed subdivisions which refine  $P_{\pm,i} := (T_{\pm}, \alpha_i)$ . If  $\sigma_i = \text{Conv}(\{e_{\alpha_j}^\vee : j \neq i\} \cup \{\varsigma_A\})$  then we observe that  $\sigma_i = \cup_{j \neq i} \sigma_{P_{\pm,i}}$  so that  $\mathcal{F}_{\Theta_p(A)}$  refines the fan  $\{\sigma_i\}$ . But the latter is simply the fan for the total space  $\mathcal{O}(-1)$  over  $\mathbb{P}^{d+1}$ .

As  $\mathcal{F}_{\Theta(A)}$  is obtained by the quotient of  $\mathcal{F}_{\Theta_p(A)}$ , we obtain the stacky fan  $\mathcal{F}_{\Theta(A)} \subset \mathbb{R}^A / \mathbb{R} \cdot (1, \dots, 1)$  as the projection. From the observations above, it is clear that the image fan for  $\mathcal{F}_{\Theta(A)}$  is a refinement of the fan for  $(d+1)$ -dimensional projective space so that there is a toric map

$$(87) \quad \varphi : \mathcal{X}_{\Theta(A)} \rightarrow \mathbb{P}^{d+1}.$$

It is also clear from the description of the refining divisors  $b_{ij}$  that this map is a stacky blowdown along the projective subspaces  $V_{ij} = \{Z_i = 0 = Z_j\}$  where  $[Z_0 : \dots : Z_{d+1}]$  are projective coordinates for  $\mathbb{P}^{d+1}$ . From the quotient description of  $\mathcal{X}_{\Theta(A)}$ , we see that the polarization  $\mathcal{O}(1)$  on  $\mathbb{P}^{d+1}$  pulls back to  $\mathbf{O}_A(1)$  as does the section  $\sum Z_i$ .  $\square$

Recall that the hypersurface  $\mathcal{Y}_A \subset \mathcal{X}_{\Theta(A)}$  is defined as the zero locus of  $s_A \in H^0(\mathcal{X}_{\Theta(A)}, \mathbf{O}_A(1))$  which implies that  $\mathcal{Y}_A$  is the proper transform of the zero locus

$$Z_0 + \dots + Z_{n+1} = 0$$

on  $\mathbb{P}^{d+1}$ .

Using the previous proposition, we easily obtain the secondary stack associated to an extended circuit. Let  $d_{\pm} = \pm \gcd\{a_i : \alpha_i \in A_{\pm}\} / |K_A|$ ,

**Proposition 4.8.** *The secondary stack of  $\mathcal{X}_{\Sigma(A)}$  is a  $K_A$  gerbe over the weighted projective line  $\mathbb{P}(d_+, d_-)$ .*

*Proof.* Recall that the stacky fan  $\Sigma_{\Sigma(A)}$  is obtained as the colimit stack of the map in diagram 32, which in this case reduces to

$$(88) \quad \begin{array}{ccc} \mathbb{Z}^{\overline{\Theta_p(A)}} & \xrightarrow{\beta_{\overline{\Theta_p(A)}}} & (\mathbb{Z}^A) \\ p_1 \downarrow & & \alpha_A^* \downarrow \\ \mathbb{Z}^2 & \xrightarrow{\beta_{\Sigma(A)}} & \mathbb{Z} \oplus \text{Ext}^1(K_A, \mathbb{Z}) \end{array}$$

where  $\alpha_A^*(e_{\alpha_i}^\vee) = (a_i, \delta(e_{\alpha_i}^\vee))$  and  $p_1(e_{\alpha_i}^\vee) = a_i/d_{\pm}$  depending on whether  $\alpha_i \in A_{\pm}$ .  $\square$

To avoid discussion of global coverings, we will restrict to the case where  $K_A = 1$  for the rest of the section. In this case, we observe that the map  $\pi : \mathcal{Y}_A \rightarrow \mathcal{X}_{\Sigma(A)}$  is the restriction of the weighted pencil  $[s_0 : s_\infty]$  on  $\mathbb{P}^{d+1}$  where

$$(89) \quad [s_0 : s_\infty] = \left[ \prod_{\alpha_i \in A_+} Z_i^{a_i/d_-} : \prod_{\alpha_i \in A_-} Z_i^{-a_i/d_+} \right].$$

The base locus of the pencil is the union  $\cup V_{ij}$  of cycles that are blown up in proposition 4.7. We note that the zero fiber of the pencil on  $\mathbb{P}^{n+1}$  is the degeneration of  $\mathcal{X}_Q$  corresponding to  $T_+$  and the fiber over infinity is the degeneration corresponding to  $T_-$ . These are both singular as stacks unless  $p = 1$  or  $q = 1$ . We write  $Deg$  for the set of critical values with normal crossing degenerations as fibers. Off of  $Deg$ ,

we may take a  $\mathbb{C}^*$  chart  $U \subset \mathbb{P}(d_-, d_+)$  and write  $\mathcal{U} = \pi^{-1}(U)$ . When restricting  $\pi$  to this substack, we obtain a more straightforward expression for  $\pi$  as the pencil

$$(90) \quad [t_0 : t_\infty] = \left[ \prod_{\alpha_j \in A_+} Z_j^{a_j} : \prod_{\alpha_i \in A_-} Z_i^{-a_i} \right].$$

This description yields the family of hypersurfaces  $\mathcal{Y}_A$  with one critical fiber over  $c_A = [\prod_{j \in A_+} a_j^{a_j} : \prod_{i \in A_-} a_i^{-a_i}]$ . One can obtain this fiber by the explicit expression for the discriminant of an extended circuit given in [46], or by direct calculation. As we will make use of this calculation later, we write out the results here.

First we order the indices of  $A$  so that  $a_i > 0$  for  $0 \leq i \leq p-1$ ,  $a_i = 0$  for  $p \leq i \leq d+3-q$  and  $a_i < 0$  for  $d+2-q \leq i \leq d+1$ . Take  $D = \{\prod_{p \leq j \leq d+3-q} Z_j = 0\}$  to be a normal crossing divisor and  $C = \cap_{p \leq j \leq d+3-q} \{Z_j = 0\}$  to be the maximal intersection of components in  $D$ . Choose the chart  $Z_0 = 1$  so that  $z_i = Z_i/Z_0$  and impose the relation  $z_{d+1} = -1 - \sum_{1 \leq i \leq d} z_i$  for  $\mathcal{Y}_A$ . Then on  $\mathcal{Y}_A^\circ$  we have independent coordinates  $\mathbf{z} = (z_1, \dots, z_d)$  and see

$$(91) \quad \tilde{\pi}(\mathbf{z}) = \left( -1 - \sum_{1 \leq i \leq d} z_i \right)^{a_{d+1}} \prod_{i=1}^d z_i^{a_i}.$$

Computing the derivative, we obtain

$$(92) \quad d\tilde{\pi} = \left( 1 + \sum_{1 \leq i \leq d} z_i \right)^{a_{d+1}-1} \prod_{i=1}^d z_i^{a_i} \left( \sum_{k=1}^d \frac{a_{d+1}z_k + a_k(1 + \sum_{1 \leq i \leq d} z_i)}{z_k} dz_k \right).$$

One observes that every  $dz_j$  coefficient is non-zero for  $j \in A_0$  so that  $d\tilde{\pi}$  satisfies condition (i) from proposition 3.15 for every  $\mathbf{z}$ . Restricting to  $C$ , we see that this is zero for all  $i$  with  $i \notin A_0$  iff  $a_j z_k = a_k z_j$  for all  $j, k \notin A_0$  and  $-a_0 z_k = (a_{d+1} + \sum_{1 \leq j \leq d} a_j) z_k = -a_k$ . These conditions imply that there is an isolated degenerate point at  $p = (a_1/a_0, \dots, a_d/a_0)$  and condition (ii) from proposition 3.15 is satisfied at  $p$ . In homogeneous coordinates we see this point is simply  $[a_0 : \dots : a_{d+1}]$  validating the above formula for  $c_A$ .

To see that  $c_A$  is a smooth point of  $\mathcal{E}_A$  in  $\mathcal{X}_{\Sigma(A)}$ , we need only verify the third condition. A simple but tedious computation yields

$$(93) \quad \text{Hess}_p(\tilde{\pi})|_{T_p C} = -c_A a_0^2 (\text{Diag}(a_1^{-1}, \dots, a_{p-1}^{-1}, a_{d+2-q}^{-1}, \dots, a_d^{-1}) + a_{d+1}^{-1} \mathbf{1}).$$

where  $\text{Diag}$  denotes the diagonal matrix and  $\mathbf{1}$  denotes the matrix with 1 in every entry. This matrix is trivially invertible leading to the following proposition:

**Proposition 4.9.** *Let  $A$  be an extended circuit of signature  $(p, q; r)$  with  $|K_A| = 1$ . The weighted pencil  $\pi : (\mathcal{Y}_A, \partial \mathcal{Y}_A) \rightarrow \mathbb{P}(d_-, d_+)$  is a  $\partial$ -framed pencil. The singular fibers at  $0, \infty$  are toric hypersurface degenerations and the singular fiber over  $c_A$  is a stratified Morse function of codimension  $r$ .*

*Proof.* The first statement follows from proposition 4.8. The description of the critical fibers over  $0, \infty$  follows from the identifications made above. Lastly, the statement on the codimension of the stratified Morse function follows from the observation that  $\partial \mathcal{Y}_A = (\cup V_{ij}) \cup (\cup_{\alpha_i \in A_0} D_i)$  where  $D_i = \{Z_i = 0\}$ . One calculates then that  $\pi$  is smooth on all intersections of the divisors  $D_i$  other than the maximal



intersection  $\cap_{\alpha_i \in A_0} D_i$ . The fact that it is non-degenerate on this intersection then follows from the case of the non-degenerate circuit.  $\square$

We write  $\pi_0 : (\mathbb{C}^*)^{d+1} \rightarrow \mathbb{C}^*$  for the restriction of  $\pi$  to the complement of the coordinate divisors on  $\mathbb{P}^{d+1}$ . We now fix a point  $t_0 \in \mathcal{X}_{\Sigma(A)}(\mathbb{R})$  near infinity and let  $\delta_0, \delta_1$  and  $\delta_\infty$  be paths around 0,  $c_A$  and  $\infty$ . Here  $\delta_1$  and  $\delta_\infty$  are straight line paths and  $\delta_0$  is a concatenation of a straight line path to an  $\varepsilon$  neighborhood of  $c_A$ , a positive semicircle around  $c_A$  and a straight line path to 0.

Our main theorem now appears as a consequence of the previous sections and discussion.

**Theorem 4.10.** *Let  $(Q, A)$  be an extended circuit with  $|K_A| = 1$ ,  $T_i = \mathbf{P}(\delta_i)$  and*

$$\mathbf{t} = \left( -\frac{2\pi \gcd(a_i, a_j)}{\text{lcm}(a_i, a_j)} : a_i > 0, a_j < 0 \right).$$

*Then*

$$(94) \quad T_0 T_1 T_\infty = \tau(\mathbf{t})$$

*in  $\pi_0(\text{Sym}^{\mathbf{F}}(\mathcal{Z}_A(t_0), \partial \mathcal{Z}_A(t_0)))$ .*

*Proof.* The only result needed for the theorem is the computation of the Chern class for the rigid boundary divisors associated to  $b_{ij}$ . For this recall that the degree of  $\mathcal{O}(-1)$  on  $\mathbb{P}(d_1, d_2)$  is  $c_1(\mathcal{O}(-1)) = -1/d_1 d_2$  [2]. In the proof of proposition 4.7, we saw that  $b_{ij} = \frac{1}{r_{b_{ij}}}(a_i e_{\alpha_j}^\vee + a_j e_{\alpha_i}^\vee)$ . The star of  $b_{ij}$  is thus the product  $\mathbb{P}\left(\frac{a_i}{r_{b_{ij}}}, \frac{a_j}{r_{b_{ij}}}\right)$  and the divisor  $D_{ij} \subset \mathcal{X}_Q$  associated to the face  $F_{ij}$ . The normal bundle restricted to an embedded  $\mathbb{P}\left(\frac{a_i}{r_{b_{ij}}}, \frac{a_j}{r_{b_{ij}}}\right)$  is clearly  $\mathcal{O}(-1)$  which implies the change of framing associated to the boundary divisor  $b_{ij}$  is  $\frac{r_{b_{ij}}^2}{a_i a_j}$  which equals the indicated factor under the assumption  $|K_A| = 1$ .  $\square$

**4.2. Vanishing cycle descriptions.** In this section, we give a full characterization of the vanishing cycle associated to a circuit. The result is a coordinate description of the vanishing thimble and cycle of the critical value  $c_A$  along with suggestive limits near the hypersurface degenerations. We start by observing a basic fact about vanishing thimbles and cycles. Suppose  $V = V_1 \oplus V_2$  is a Hermitian vector space and the sum is orthogonal. Let  $f_i : V_i \rightarrow \mathbb{C}$  be two functions with Morse-Bott singularity over the origin. Let  $\mathcal{T}_i$  be the vanishing thimble over the positive real numbers and  $\mathcal{V}_i(t)$  the vanishing cycle over  $t$ .

**Lemma 4.11.** *The vanishing thimble of  $f = f_1 + f_2$  is the sum  $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ . The vanishing cycle is the join:*

$$(95) \quad \mathcal{V}(t) = \cup_{0 \leq s \leq t} \mathcal{V}(s) \oplus \mathcal{V}(t-s)$$

*Proof.* The lift of  $-\partial_z$  can be written as the normalized linear sum of the lifts to  $V_1$  and  $V_2$ . This implies that any solution is a normalized combination of solutions, so that with initial conditions in  $\mathcal{T}_1 + \mathcal{T}_2$ , we must have a flow to 0 and conversely, any flow line without such initial conditions must not converge to 0. The second assertion follows immediately from the first.  $\square$

We now give a detailed description of the vanishing thimble of  $\pi_0$  up to isotopy. To do this, we choose the torus invariant symplectic form on  $(\mathbb{C}^*)^{d+1} \subset (\mathbb{C}^*)^{d+2} \rightarrow \mathbb{P}^{d+1}$  given by

$$(96) \quad \tilde{\omega} = \frac{i}{2} \left( \sum_{i=0}^{d+1} \frac{dZ_i \wedge d\bar{Z}_i}{|Z_i|^2} \right),$$

where the map above is the quotient by the diagonal action. We will work with logarithmic coordinates  $X_i$  where  $(Z_0, \dots, Z_{d+1}) = \phi(X_0, \dots, X_{d+1})$  is given by

$$(97) \quad \phi(X_0, \dots, X_{d+1}) = (a_0 e^{X_0}, \dots, a_{d+1} e^{X_{d+1}}).$$

Then there is a commutative diagram

$$(98) \quad \begin{array}{ccc} \mathbb{C}^{d+2} & \xrightarrow{\langle \mathbf{a}, - \rangle} & \mathbb{C} \\ \downarrow \phi & & \downarrow c_A \exp(-) \\ \mathbb{C} & \xleftarrow{s_A} (\mathbb{C}^*)^{d+2} \xrightarrow{\pi_0} & \mathbb{C}^* \end{array}$$

and the composition  $\tilde{s}_A = s_A \circ \phi$  is the constraint defining the logarithm of the hypersurface. Finally, there is the homogeneous diagonal action on  $(\mathbb{C}^*)^{d+2}$  that arises in the space  $\mathbb{C}^{d+2}$  as adding  $(t, \dots, t)$ . Identify  $\mathcal{Y}_A^\circ$  with its logarithmic coordinates in the subset  $\{\tilde{s}_A = 0 = \sum X_i\}$ . Our aim is then to find the vanishing thimbles and cycles for  $\langle \mathbf{a}, - \rangle$  on  $\mathcal{Y}_A^\circ$ .

We achieve this by explicit computation. Let

$$(99) \quad \kappa(\theta) := \log(\theta \csc(\theta)) + i\theta$$

and observe that  $\{\kappa(\theta) : -\pi < \theta < \pi\}$  and  $\mathbb{R} \subset \mathbb{C}$  are the positive and negative vanishing thimbles for  $f(z) = z + 1 - e^z$  and any positive real multiple thereof. Define the function  $F : \mathbb{C}^{d+1} \rightarrow \mathbb{C}$ ,

$$(100) \quad F(X_0, \dots, X_{d+1}) = \sum a_i X_i - \sum a_i e^{X_i}.$$

We reorder the indices of  $\mathbf{a}$  so that  $a_i < 0$  for all  $0 \leq i \leq p-1$  and  $a_i > 0$  otherwise. By repeated application of lemma 4.11, we may give coordinates to the vanishing thimbles of  $F$  via

$$(101) \quad \tilde{\mathcal{T}}_+ = \{(r_0, \dots, r_{p-1}, \kappa(\theta_p), \dots, \kappa(\theta_{d+1})) : r_i \in \mathbb{R}, -\pi < \theta_i < \pi\},$$

$$(102) \quad \tilde{\mathcal{T}}_- = \{(\kappa(\theta_0), \dots, \kappa(\theta_{p-1}), r_p, \dots, r_{d+1}) : r_i \in \mathbb{R}, -\pi < \theta_i < \pi\}.$$

We take the coordinates  $r_i$  and  $\theta_i$  for the thimbles and define real codimension 2 subspaces of  $\mathcal{T}'_\pm \subset \tilde{\mathcal{T}}_\pm$  as the zero loci of the functions  $g_\pm$  and  $h_\pm$  respectively where

$$(103) \quad g_+(r_0, \dots, r_{p-1}, \theta_p, \dots, \theta_{d+1}) = \sum_{j=p}^{d+1} a_j \theta_j,$$

$$(104) \quad h_+(r_0, \dots, r_{p-1}, \theta_p, \dots, \theta_{d+1}) = \sum_{j=0}^{p-1} a_j e^{r_j} + \sum_{j=p}^{d+1} a_j \theta_j \cot(\theta_j),$$

while the functions  $g_-$ ,  $h_-$  are these functions with the indices  $0 \leq i < p$  and  $p \leq j \leq (d+1)$  switched.

A straightforward calculation shows that  $\mathcal{T}'_{\pm}$  are smooth submanifolds of  $\tilde{\mathcal{T}}_{\pm}$ . We take  $\mathcal{T}_{\pm}$  to be the image of  $\mathcal{T}'$  in  $\mathbb{C}^{d+1}$  which maps as a universal cover to  $(\mathbb{C}^*)^{d+1} \subset \mathbb{P}^{d+1}$ . Let  $\tilde{\pi} : \mathcal{Y}_A^{\circ} \rightarrow \mathbb{C}$  via  $\tilde{\pi}(X_0, \dots, X_{d+1}) = \sum_{i=0}^{d+1} a_i X_i$ .

**Proposition 4.12.** *The positive and negative vanishing thimbles of  $\tilde{\pi}$  on  $\mathcal{Y}_A^{\circ}$  are fiberwise Lagrangian isotopic to  $\mathcal{T}_{\pm}$ .*

*Proof.* As the cases are symmetric, we will prove this for  $\mathcal{T}_{+}$ . We first observe that  $\mathcal{T}'_{+}$  is smooth and that adding  $\{(t, \dots, t) : t \in \mathbb{C}\}$  gives a cylinder over  $\mathcal{T}'_{+}$ . Indeed, if we add  $(t, \dots, t)$  to any point in  $\mathcal{T}'_{+}$ , we see that we displace it completely, since  $\log(\theta \csc(\theta))$  is an injective function of  $\theta$ . This implies that  $\mathcal{T}'_{+}$  maps bijectively to the quotient  $\mathcal{T}_{\pm}$  by the homogeneous action, so for simplicity we work with  $\mathcal{T}'_{+}$  instead.

Given that  $\mathcal{T}'_{+} + \{(t, \dots, t) : t \in \mathbb{R}\}$  is a Lagrangian submanifold, invariant under the action giving the symplectic quotient  $\mathbb{C}^{d+2}/\mathbb{C}$ , we have that  $\mathcal{T}'_{+}$  projects to a smooth Lagrangian. Another point that follows from this is that, since  $h_{+} = 0 = g_{+}$ , we have  $\text{im}(\tilde{\pi}|_{\mathcal{T}'_{+}}) = \text{im}(F|_{\mathcal{T}'_{+}}) \subset \text{im}(F|_{\mathcal{T}_{+}}) = \mathbb{R}_{\geq 0}$ . The isotopy claim then follows from a check that the function  $\tilde{\pi}$  has smooth fibers when restricted to  $\mathcal{T}'_{+}$  everywhere but 0 where it achieves a global minimum. We calculate  $\eta = dk \wedge dg_{+} \wedge dh_{+}$  to be

$$\begin{aligned} \eta &= \left( \sum_{0 \leq i \leq p-1 < j \leq d+1} a_i a_j dr_i \wedge d\theta_j \right) \wedge dh_{+}, \\ &= \sum_{0 \leq i < k \leq p-1 < j \leq d+1} a_i a_j a_k (e^{r_k} - e^{r_i}) dr_k \wedge dr_i \wedge d\theta_j \\ &\quad + \sum_{0 \leq i \leq p-1 < j < k \leq d+1} a_i a_j a_k \left( \frac{\sin(2\theta_j) - 2\theta_j}{2\theta_j \sin(\theta_j)} - \frac{\sin(2\theta_k) - 2\theta_k}{2\theta_k \sin(\theta_k)} \right) dr_i \wedge d\theta_j \wedge d\theta_k. \end{aligned}$$

It is an easy check to see that  $(\sin(2\theta) - 2\theta)/2\theta \sin(\theta)$  is a strictly decreasing function. This implies that  $dk \wedge dg_{+} \wedge dh_{+} = 0$  if and only if  $\theta_j = \theta_k$  and  $r_i = r_k$  for all applicable  $i, j, k$ . Since  $g_{+} = 0$  though, this implies that  $\theta_j = 0$  for all  $j$ . And then from  $h_{+} = 0$ , we obtain  $r_i = 0$  for all  $i$ . Finally, since  $\tilde{\pi}$  has positive definite Hessian on  $\mathcal{T}_{+}$  and  $\mathcal{T}'_{+}$  is a smooth submanifold, we have that it has positive definite Hessian on  $\mathcal{T}'_{+}$  so that this is a Morse minimum.  $\square$

Having obtained an implicit coordinate description of the vanishing thimbles, we proceed to describe their intersection with real loci in  $\mathcal{Y}_A$ . In the next subsection, we will also use this, along with matching path arguments, to obtain the location of the vanishing cycles in dimension 1. Indeed, one motivation for the last proposition was to open up a detailed analysis on the Fukaya category and Floer theory of circuits in general.

We take the set of  $A$ -positive real elements of  $\mathbb{R}\mathbb{P}^{d+1} \subset \mathbb{C}\mathbb{P}^{d+1}$  to be

$$(105) \quad \mathcal{Y}_A^{+}(\mathbb{R}) = \{[r_0 : \dots : r_{d+1}] \in \mathcal{Y}_A : r_i r_j \geq 0 \text{ iff } a_i a_j \geq 0, r_k = 0 \text{ iff } a_k = 0\}.$$

In the degenerate case  $r \neq 0$ , one sees that  $\mathcal{Y}_A^{+}(\mathbb{R})$  is a real manifold with codimension  $r$  corners. For any such real  $n$ -dimensional manifold  $M$ , we may define a real stratified Morse function  $f : M \rightarrow \mathbb{R}$  to be any function for which, near a

degenerate point  $p$ , there exists a chart  $U \subset \mathbb{R}^k \times \mathbb{R}_{\geq 0}^{n-k+1}$  around  $p$  for which

$$(106) \quad f(x_1, \dots, x_n) = \sum_{i=1}^t x_i^2 - \sum_{i=t+1}^k x_i^2 + \sum_{i=n-k+1}^n x_i.$$

Near such a point, we say that  $f$  has signature  $(t, k-t; n-k)$  (see [29]). We now prove:

**Proposition 4.13.** *Suppose  $A$  is an extended circuit with signature  $\sigma_A = (p, q; r)$ . The function  $(-1)^q \tilde{\pi}$  restricted to  $\mathcal{Y}_A^+(\mathbb{R})$  is a real stratified Morse function of signature  $(q-1, p-1; r)$ .*

*Proof.* We start by observing that the real form of proposition 3.15 also holds, but the signature of the Hessian must be calculated in this case in order to obtain an equivalence up to diffeomorphism. The computations 92, 93 are the same, up to a multiplicative factor of  $(-1)^q$ , for  $\tilde{\pi}$  on  $\mathcal{Y}_A^+(\mathbb{R})$  as it has the same form and constraints with respect to the variables  $(r_1, \dots, r_d)$ . Thus, we need only compute the signature of  $\tilde{H} := (-1)^q \text{Hess}_p(\tilde{\pi})|_C$  on  $\mathcal{Y}_A^+(\mathbb{R})$ . Letting  $V_+ = \text{Lin}_{\mathbb{R}}\{e_i : d+2-q \leq i \leq d\}$  it is easy to see that  $\tilde{H}$  restricts to a positive definite form on  $V_+$ . On the other hand, were we to have chosen  $\{Z_{d+1} = 1\}$  and  $z_0 = -1 - \sum z_i$ , the form of the Hessian would have switched to

$$(107) \quad \tilde{H}' = \text{Hess}_p(\tilde{\pi} \circ \phi)|_{T_p C} = -c_A a_{d+1}^2 (\text{Diag}(a_1^{-1}, \dots, a_{p-1}^{-1}, a_{d+2-q}^{-1}, \dots, a_d^{-1}) + a_0^{-1} \mathbf{1}).$$

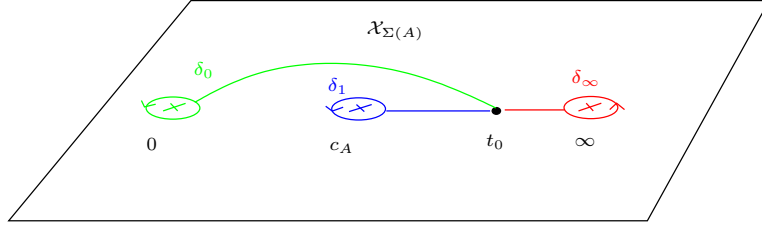
Taking  $V_- = \text{Lin}_{\mathbb{R}}(\{e_i : 1 \leq i \leq p-1\})$ , it is easy to see that  $\tilde{H}'$  is negative definite on  $V_-$ . As the signature is independent of the choice of coordinates, this gives the claim.  $\square$

This proposition can be thought of as a mild extension of the  $(p, q)$  surgery observations in [46]. Following an approach by Viro [58], they observed that the real loci of  $\mathcal{Y}_A^+(\mathbb{R})$  could be combinatorially understood through a cover of the tropical hypersurface. A circuit modification then was found to be a  $(p, q)$  surgery. Our result gives this surgery as a cell modification associated to the Morse function  $\pi : \mathcal{Y}_A^+(\mathbb{R}) \rightarrow \mathbb{R}$ .

**4.3. Examples in dimension 1.** In this section we illustrate three examples in dimension 1 of the circuit relation in full detail. The first relation is known as the lantern relation for mapping class groups of marked curves and, to a large degree, is the case that inspired this paper. The next example yields the star relation. We observe that the circuit stack in this example, as well as its higher dimensional generalizations, arises naturally in the context of homological mirror symmetry. We refer to [25] for general background on mapping class groups of marked curves and classical proofs of these relations.

For every example, we take a fiber  $t_0 \in \mathbb{R}_{>1}$  near  $\infty$  and choose the distinguished basis of paths  $\delta_0, \delta_1$  and  $\delta_\infty$  on  $\mathcal{X}_{\Sigma(A)}$  as in theorem 4.10 and figure 4.

**4.3.1. Circuit of signature  $(2, 2)$ .** Here we take the set  $A = \{(0, 0), (1, 0), (1, 1), (0, 1)\}$  and fix the orientation of  $A$  as  $\mathbf{a} = (1, -1, 1, -1)$ . We have that  $\mathcal{Y}_A = \{X_0 + X_1 + X_2 + X_3 = 0\} \subset \mathbb{P}^3$  and  $\pi$  is defined as the pencil  $[X_0 X_2 : X_1 X_3]$ . Every fiber  $\mathcal{Z}_A(t)$  for  $t \in \mathbb{C}^* - \{1\}$  is isomorphic to  $\mathbb{P}^1$  with boundary divisor consisting of four


 FIGURE 4. Distinguished basis on  $\mathcal{X}_{\Sigma(A)}$ 

points given as an intersection with  $\cup_{i=0}^3 D_i$  where  $D_i = \{X_i = 0 = X_{i+1}\}$  using an index in  $\mathbb{Z}/4\mathbb{Z}$ . We give coordinates to any smooth fiber

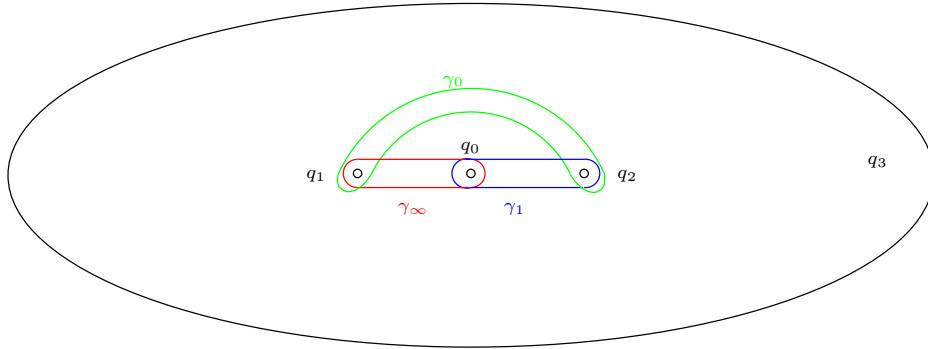
$$\mathcal{Z}_A(t) = \{[1 - tx : (tx - 1)x : tx(1 - x) : x - 1] : x \in \mathbb{C}\}$$

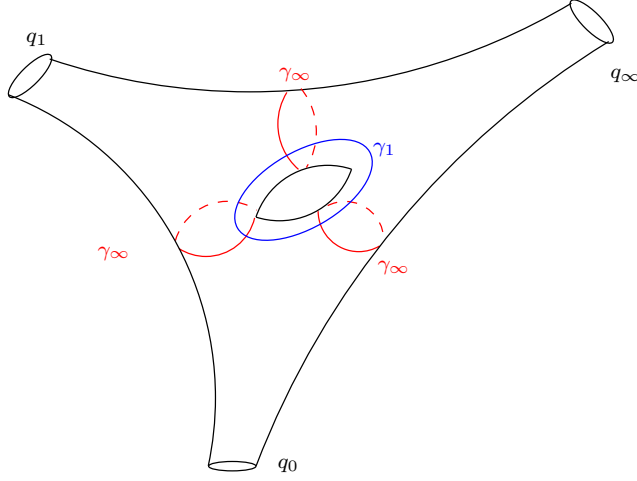
with boundary divisor,

$$\begin{aligned} q_0 &= D_0 \cap \mathcal{Z}_A(t) = \{x = t^{-1}\}, \\ q_1 &= D_1 \cap \mathcal{Z}_A(t) = \{x = 0\}, \\ q_2 &= D_2 \cap \mathcal{Z}_A(t) = \{x = 1\}, \\ q_3 &= D_3 \cap \mathcal{Z}_A(t) = \{x = \infty\}. \end{aligned}$$

Over the limiting degeneration values of  $t = 0$  and  $\infty$ , one sees that this converges to give parameterizations of the intersections  $\{X_2 = 0\} \cap \mathcal{Y}_A$  and  $\{X_3 = 0\} \cap \mathcal{Y}_A$ , respectively.

As  $t_0$  is close to  $\infty$ , we have that  $q_0 > 0$  is close to zero and indeed tends to  $q_1$ . This reflects the bubbling of the intersection  $\mathcal{Y}_A \cap \{X_0 = 0\}$  off in the limit and we see that the vanishing cycle of  $\delta_\infty$  is a loop  $\gamma_\infty$  around  $t_0$  and 0 in the  $x$ -plane. In a similar vein, we may follow the path  $\delta_1$  from  $t_0$  to 1 and observe that the point  $q_0$  follows the straight line path to  $q_2$ . Thus the vanishing cycle associated to  $\delta_1$  is  $\gamma_1$ . Finally, tracing the path  $\delta_0$ , we observe  $q_0$  going under  $q_2$  and towards  $q_3$ . The vanishing cycle may be pulled back along this path and is seen to be equivalent to  $\gamma_0$  which, up to isotopy, is illustrated in figure 5.


 FIGURE 5. The  $(2, 2)$  circuit relation or the lantern relation

FIGURE 6. The  $(1,3)$  circuit relation or the star relation

We also note that by implementing proposition 4.12 of the last section, we can obtain an implicit equation for  $\gamma_1 \subset \mathcal{Z}_A(t)$ ,

$$\gamma_1 = \left\{ x + iy \in \mathbb{C} : (x-1)^2 + y^2 = 1 - \frac{1}{t} \right\}.$$

It is easy to verify that this formula agrees with the description given above and in figure 5. Applying theorem 4.10 in this example yields the well known lantern relation.

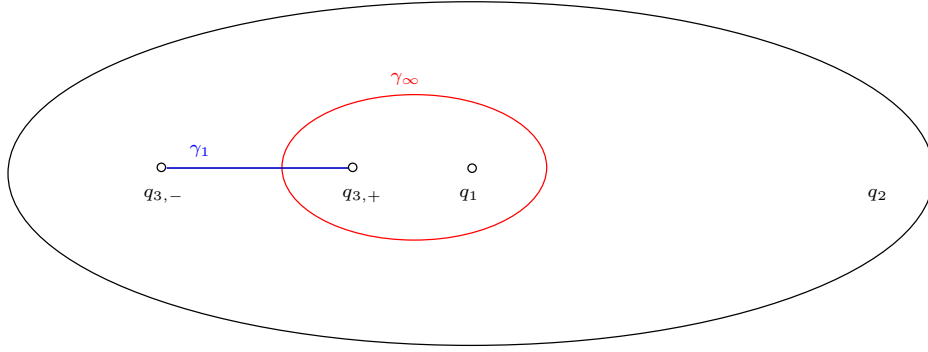
**4.3.2. Circuit of signature  $(1,3)$ .** In our example of a  $(1,3)$  circuit, we take the set  $A = \{(0,0), (1,0), (0,1), (-1,-1)\}$  and fix  $\mathbf{a} = (3, -1, -1, -1)$ . We have the same hypersurface  $\mathcal{Y}_A \subset \mathbb{P}^3$  as before, but with  $\tilde{\pi}([X_0 : X_1 : X_2 : X_3]) = [X_0^3 : X_1 X_2 X_3]$ . The fibers  $\mathcal{Z}_A(t)$  of  $\tilde{\pi}$  are elliptic curves with boundary points indexed by the divisors  $D_i = \{X_i = 0 = X_0\} = \{q_i\}$  for  $i = 1, 2, 3$ . Near  $t = \infty$ , we have  $\mathcal{Z}_A(t)$  approaches an intersection of  $\mathcal{Y}_A$  with the three divisors in  $\{X_1 = 0\}$ ,  $\{X_2 = 0\}$  and  $\{X_3 = 0\}$  which subdivides it into three pairs of pants. By proposition 4.13 we have that the vanishing cycle associated to the critical value  $c_A$  converges to the  $A$ -positive real locus  $\mathcal{Y}_A(\mathbb{R})$  intersected with the three pairs of pants. These facts are illustrated in figure 6. The circuit relation in this example is the star relation.

In more generality, we may consider the signature  $(1, d+1)$  case with  $\mathbf{a} = (v_A, -a_1, \dots, -a_{d+1})$ . Again we have that the hypersurface  $\mathcal{Y}_A$  is  $\{\sum_{i=0}^{d+1} X_i = 0\}$  in  $\mathbb{P}^{d+1}$  and

$$\tilde{\pi}([X_0 : \dots : X_{d+1}]) = [X_0^{v_A} : X_1^{a_1} \dots X_{d+1}^{a_{d+1}}].$$

Here,  $\mathcal{X}_{\Sigma(A)} = \mathbb{P}(v_A, 1)$  and that we may take the chart around zero to be the map  $z^{a_0}$ . Pulling  $\tilde{\pi}$  back along this atlas we obtain a map  $w : (\mathbb{C}^*)^d \rightarrow \mathbb{C}$ . Indeed, we take  $\tilde{\pi} = [t^{v_A} : 1]$  which after restricting to  $X_1^{a_1} \dots X_{d+1}^{a_{d+1}} = 1$  gives  $X_0 = t$  so that the function

$$w(X_1, \dots, X_d) = X_0 = - \sum_{i=1}^d X_i - \frac{1}{X_1^{a_1/a_{d+1}} \dots X_d^{a_d/a_{d+1}}}.$$

FIGURE 7. The  $(1, 2; 1)$  circuit relation

Referring to [33] we have that, up to a scale, the map  $\pi$  is the equivariant quotient of the homological mirror LG model of the weighted projective space  $\mathbb{P}(a_1, \dots, a_{d+1})$ . This will appear again as one piece of a general conjectural program for homological mirror symmetry in the final section.

4.3.3. *Circuit of signature  $(1, 2; 1)$ .* In our only degenerate example, we observe a relation between braids and loops. We take  $A = \{(0, 0), (1, 0), (-1, 0), (0, 1)\}$ ,  $\mathbf{a} = (2, -1, -1, 0)$ . Here  $\mathcal{X}_{\Theta(A)}$  is the blow-up of  $\mathbb{P}^3$  along the two coordinate lines  $L_1 = \{X_0 = 0 = X_1\}$  and  $L_2 = \{X_0 = 0 = X_2\}$  which are the base locus of the pencil  $\tilde{\pi}$  given as

$$(108) \quad \tilde{\pi}([X_0 : X_1 : X_2 : X_3]) = [X_0^2 : X_1 X_2].$$

Since  $A$  is a degenerate circuit, the divisor  $\{X_3 = 0\}$  is not contained in a fiber over 0 or infinity, but rather intersects  $\mathcal{Z}_A(t)$  in two points everywhere except over the degenerate point  $[2 : -1 : -1 : 0]$  with value  $c_A = 4$ . We give  $\mathcal{Z}_A(t)$  coordinates,

$$(109) \quad \mathcal{Z}_A(t) = \{[tx : x^2 : t : -tx - x^2 - t] : x \in \mathbb{C}\}.$$

The boundary points on  $\mathcal{Z}_A(t)$  are then,

$$\begin{aligned} q_1 &= \mathcal{Z}_A(t) \cap \{X_1 = 0\} = \{x = 0\}, \\ q_2 &= \mathcal{Z}_A(t) \cap \{X_2 = 0\} = \{tx = \infty\}, \\ q_{3,\pm} &= \{x = -t \pm \sqrt{t^2 - 4t/2}\}. \end{aligned}$$

As  $t$  tends from  $c_A$  to  $t_0$ , we see that  $q_{3,\pm}$  splits along the real axis. By proposition 4.13, we have that  $\mathcal{Y}_A(\mathbb{R})$  gives the positive vanishing cycle, implying that the vanishing cycle  $\gamma_1$  for  $\delta_1$  is the interval  $x \in [q_{3,-}, q_{3,+}]$ . Tending from  $t_0$  to  $\infty$ , one observes  $q_{3,+}$  converging to  $-1$  and  $q_{3,-}$  bubbling off with  $\infty$ . Thus we may draw a vanishing cycle  $\gamma_\infty$  around  $\infty$  and  $q_{3,-}$  corresponding to  $\delta_\infty$ .

At  $t = 0$  we have a  $\mathbb{Z}/2\mathbb{Z}$  orbifold point where, in the coordinates given by  $x$ , we have quotiented by the action. Ignoring the framing on the endpoints  $q_{3,\pm}$  of the braid and letting  $T_{\partial\mathcal{Z}_A(t_0)}$  be a full twist about  $q_1$  and  $q_2$ , we have the relation  $(T_1 T_\infty)^2 = T_{\partial\mathcal{Z}_A(t_0)}$ . This does not seem to have a direct analog in the literature, but can be thought of as a hyperelliptic relation for a braid and a loop.

**4.4. Regeneration.** In contrast to the topology of discriminant complements (see [22]), the geometry of the principal  $A$ -determinant complement seems relatively unexplored. For extended circuits, we have seen that the project of understanding  $\mathcal{X}_{\Sigma(A)} - E_A$  is easily completed as all such spaces are  $K(\pi, 1)$  spaces. On the other hand, as one considers more complicated sets  $A$ , the complexity of the topology of their determinant complements grows rapidly. In order to retain the information obtained from more basic cases of  $A' \subset A$  such as circuits, we require a method of regeneration. In large measure, the toric and symplectic preliminaries of the first two sections were designed to make such a method possible and accessible.

Let  $A \subset \mathbb{Z}^d$  and  $A' \subset A$  be finite subsets and  $S = \{(Q_i, A_i) : i \in I\}$  a regular subdivision of  $Q$  such that  $(Q_i, A_i)$  is a marked simplex for all  $A_i$  which does not contain  $A'$  and  $A_i$  is a simplicial extension of  $A'$  otherwise. Call such a subdivision a triangular extension of  $A'$ . Then there is a natural inclusion  $i_S : \mathcal{X}_{\Sigma(A')} \rightarrow \mathcal{X}_{\Sigma(A)}$  induced by the subdivision. Take  $\mathcal{X}_{\Sigma(A')}^\circ$  to be the dense torus of  $\mathcal{X}_{\Sigma(A')}$  and  $E_A^\circ$  to be the intersection of  $E_A$  with  $\mathcal{X}_{\Sigma(A')}^\circ$ . Given  $\varepsilon > 0$ , let  $\mathcal{I}_{A'}^\varepsilon \subset \mathcal{X}_{\Sigma(A')}^\circ - E_{A'}^\circ$  be the complement of an  $\varepsilon$  neighborhood of  $E_{A'}^\circ$ . For sufficiently small  $\varepsilon$ ,  $\mathcal{I}_{A'}^\varepsilon$  is diffeomorphic to  $\mathcal{X}_{\Sigma(A')}^\circ - E_{A'}^\circ$ .

**Definition 4.14.** Let  $D \subset \mathbb{C}$  be a neighborhood of the origin. A regeneration of  $\mathcal{X}_{\Sigma(A')}$  relative to  $\mathcal{X}_{\Sigma(A)}$  is a pair  $(\mathcal{I}, \psi)$  where  $\psi_- : D \times \mathcal{I} \rightarrow \mathcal{X}_{\Sigma(A)}$  is étale onto its image with  $\psi_0 : \mathcal{I} \rightarrow i_S(\mathcal{I}_{A'}^\varepsilon)$  and  $\psi_t : \mathcal{I} \rightarrow \mathcal{X}_{\Sigma(A)}$  for all  $t \neq 0$ .

The following proposition follows from theorem 2.19 and [46], section 10.1.12.

**Proposition 4.15.** *Let  $G$  be the relative isotropy subgroup of a smooth point in  $i_S(\mathcal{X}_{\Sigma(A')})$  and  $H$  a cyclic subgroup of  $G$ . Then there exists a regeneration  $(\mathcal{I}, \psi)$  of  $A'$  with  $\psi_0$  an  $H$ -cover.*

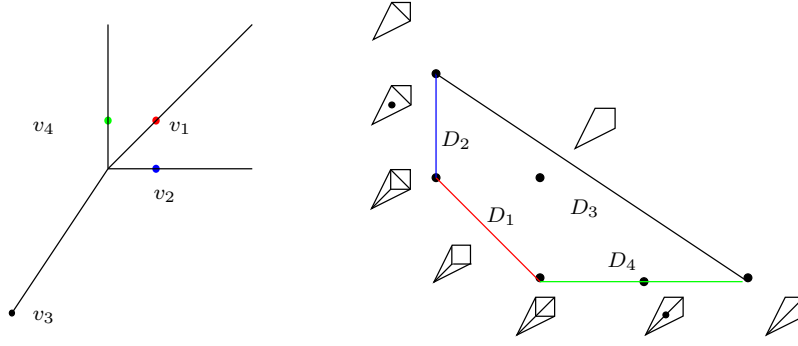
Indeed, by choosing a single interior lattice point of the cone  $C_S$  dual to the strata of  $S$ , we obtain such a regeneration. Using such a choice, one observes that  $\psi_-(p) : D \rightarrow \mathcal{X}_{\Sigma(A)}$  pulls back a toric hypersurface degeneration for every  $p \in \mathcal{I}$ . Recall from proposition 3.13 that to any toric hypersurface degeneration, such as the one associated to  $S$ , we have a decomposition  $\mathcal{Z}_\eta(t) = \cup_{i \in I} V_i$  such that  $V_i \approx \mathcal{Z}_i(0) - \partial \mathcal{Z}_i(0)$  of the nearby fibers. The covering group  $H$  is generated by the monodromy group. By varying the family of degenerations along  $\mathcal{I}$  and utilizing the description of degeneration monodromy, the group  $H$  acts on each  $V_i$  so that we have a homomorphism  $\xi_i : H \rightarrow \text{Symp}(V_i, \partial V_i) \rightarrow \text{Symp}(\mathcal{Z}_i(0))$ . This argument gives the following proposition.

**Proposition 4.16.** *Let  $(\mathcal{I}, \psi)$  be a regeneration of  $A'$  relative to  $A$ . Let  $\mathbf{Symp}_H$  be the category of standard symplectic stacks with  $H$ -action. Then there is a commutative diagram of functors:*

$$(110) \quad \begin{array}{ccc} \Pi(\mathcal{I}) & \xrightarrow{\mathbf{P}} & \mathbf{Symp}_H \\ \psi_0 \downarrow & & [-/H] \downarrow \\ \Pi(\mathcal{X}_{\Sigma(A')}) & \xrightarrow{\mathbf{P}} & \mathbf{Symp} \end{array}$$

The propositions above suggest a general method of approaching the symplectomorphism group of a toric hypersurface through an analysis of the groups on degenerated pieces. Of course, the general case of  $A$  is exceptionally complex as




 FIGURE 8. The secondary fan and polytope of  $A$ 

it requires an understanding of groups for all smaller sets  $A' \subset A$ . In this section we will see to what extent this approach is accessible in an example where  $A$  is minimally more complicated, namely  $A$  contains  $(d+3)$  points.

The general case of  $(d+3)$  points has been studied and explicit formulas for  $E_A$  are known [21]. At this level of generality, the formulas do not immediately render the geometry of the principal  $A$ -determinant or its complement accessible. However, it is worth mentioning that the discriminant component is always a rational curve in a  $\mathcal{X}_{\Sigma(A)}$ , usually with complicated singularities [34]. Our example is

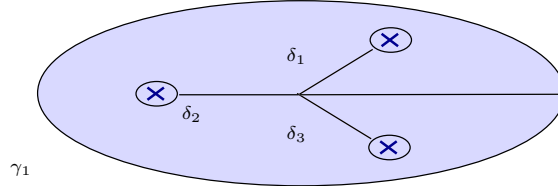
$$(111) \quad A = \{(1, 0), (0, 1), (1, 1), (-1, -1), (0, 0)\}.$$

Any non-degenerate hypersurface  $\mathcal{Z}_A(p)$  is an elliptic curve with 4 boundary points. A short calculation gives that the secondary stack has stacky fan given by

$$(112) \quad \mathcal{F}_{\Sigma(A)} = \{v_1, \dots, v_4\} = \{(1, 1), (0, 1), (-2, -3), (1, 0)\} \subset \mathbb{Z}^2.$$

The secondary fan and polytope are illustrated in figure 8.

One observes that there are four circuits  $\{C_1, C_2, C_3, C_4\}$  contained in  $A$  corresponding to the four divisors. To each circuit  $C_i$  there is a unique triangular extension given by the subdivision associated to the facet defined by  $v_i$ . Note that there are five extended circuits contained in  $A$  as the divisor  $D_4$  corresponds to the degenerate circuit supporting two extended circuits. Let us first examine regenerations of  $\mathcal{X}_{\Sigma(C_1)} = D_1$  and  $\mathcal{X}_{\Sigma(C_2)} = D_2$ . Starting with  $C_1$  we observe that  $N_{\mathcal{X}_{\Sigma(A)}} \mathcal{X}_{\Sigma(C_1)}$  is isomorphic to  $\mathcal{O}(-1)$  over  $\mathbb{P}^1$ , but the principal  $A$ -determinant vanishes on the intersections with  $D_2$  and  $D_4$ . So  $\mathcal{I}_{C_1}^\varepsilon \subset \text{orb}\{v_1\} \subset D_1$  and the normal bundle in  $\mathcal{X}_{\Sigma(A)}$  trivializes  $N_{\mathcal{X}_{\Sigma(A)}} \mathcal{I}_{C_1}^\varepsilon \approx \mathbb{C} \times \mathcal{I}_{C_1}^\varepsilon$ . Any such trivialization


 FIGURE 9. Paths for the regenerated circuit of  $C_2$

gives a regeneration. Proposition 4.16 gives us that the regenerated circuit relation is simply the  $(2, 2)$  circuit relation restricted to the region  $V_1 \subset \mathcal{Z}_A(p)$ .

The divisor  $D_2$  is  $\mathbb{P}(1, 3)$  with normal bundle  $\mathcal{O}(-1)$ . Even after deleting the point at infinity lying in  $E_A$ , we can not regenerate  $D_1$  using sections of this bundle because of the smooth stacky point at the origin, so we must consider a covering. There is only one non-trivial covering in this case, namely the étale cover  $z^3$  of  $\mathcal{I}_{C_2}^\varepsilon \subset \mathbb{P}(1, 3) - D_1 \cap D_2 \approx \mathbb{C}/\mu_3$ . To find the regeneration which extends this cover, one simply takes the stacky chart of a neighborhood  $U$  of the point  $D_1 \cap D_3$  which is  $\mathbb{C}^2/\mu_3$  where  $\zeta(t, x) = (\zeta^{-1}t, \zeta x)$ . The map  $\psi : \mathbb{C}^2 \rightarrow U$  is obviously étale and at  $t = 0$  gives the covering above, so restricting  $\psi$  to  $\psi^{-1}(U - V)$  where  $V$  is an  $\varepsilon$  neighborhood of  $E_A$  gives a regeneration of  $C_2$ . Applying proposition 4.16 to this situation, we observe that  $\psi^{-1}(U - V) \cap \{t\} \times \mathbb{C}$  is a disc with three discs removed near the third roots of unity as in figure 9. Using the proposition and theorem 4.10, composing the parallel transport  $T_i$  along the three paths  $\delta_i$  gives the cube of parallel transport along  $\gamma$  along with a full boundary twist. Taking the composition of these two operations as  $T_4$  we write simply  $T_1 T_2 T_3 = T_4$  and observe this as a relation in  $\mathcal{Z}_A(p)$ .

For what follows, we compute in the homogeneous coordinate ring  $\mathbb{C}[x_1, x_2, x_3, x_4]$  of  $\mathcal{X}_{\Sigma(A)}$  where  $\deg(x_1) = (1, 2)$ ,  $\deg(x_2) = (1, 0)$ ,  $\deg(x_3) = (1, 1)$  and  $\deg(x_4) = (2, 1)$ . Utilizing results in [46] and [21], one can calculate that  $E_A^s \cdot D_1 = 1 = E_A^s \cdot D_2$ .

One can often regenerate several subsets of  $A$  simultaneously, thereby incorporating the symplectomorphisms of the regenerated pieces into those of the hypersurface  $\mathcal{Z}_A(t)$ . We give a more systematic account of this method in the next section for extended circuits, but for now we consider sections of the ample line bundle  $\mathcal{L} = \mathcal{O}(D_1 + 3D_2)$  on  $\mathcal{X}_{\Sigma(A)}$ . Consider the pencil  $f(x_1, x_2, x_3, x_4) = [s_0 : s_\infty] :=$

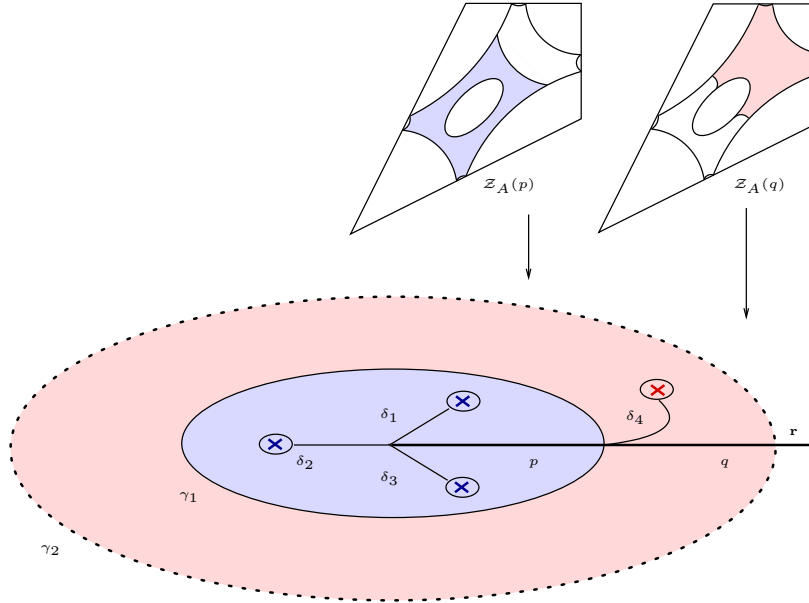


FIGURE 10. Generating paths for  $\mathbf{G}_{C_t}$  with trivialized fiber over  $\mathbf{r}$

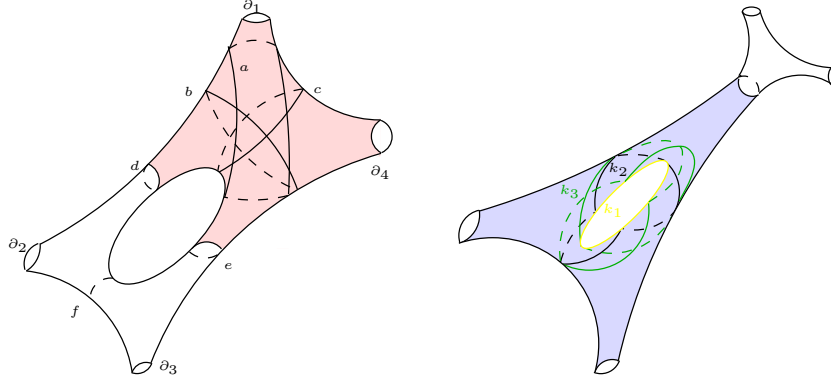


FIGURE 11.

$[x_1 x_2^3 : x_4^2]$ . Taking  $\mathcal{C}_t = \{s_0 - t s_\infty\}$ , one observes that for small  $t$ , we obtain a smooth curve which approximates  $D_1 + 3D_2$ . We wish to understand the  $\mathcal{C}_t$  subgroup  $\mathbf{G}_{\mathcal{C}_t} \subset \text{Symp}(\mathcal{Z}_A(p), \partial \mathcal{Z}_A(p))$  from definition 3.26 by viewing  $\mathcal{C}_t$  as a simultaneous regeneration of  $C_1$  and  $C_2$ . We trivialize the fibers  $\mathcal{Z}_A(p)$  along the ray  $\mathbf{r}$  and consider parallel transport  $\{T_1, \dots, T_4, \tilde{T}_1, \tilde{T}_2\}$  along the paths  $\{\delta_1, \delta_2, \delta_3, \delta_4, \gamma_1, \gamma_2\}$  as in figure 10.

Utilizing proposition 4.16, the symplectomorphisms  $T_i$  associated to the paths  $\delta_i$  can be written as compositions of disjoint Dehn twists as

$$\begin{aligned} T_1 &= T_{k_1}, \\ T_2 &= T_{k_2}, \\ T_3 &= T_{k_3}, \\ T_4 &= T_a, \\ \tilde{T}_1 &= T_b T_d^3 T_e^3 T_f^3, \\ \tilde{T}_2 &= T_c T_d^2 T_e^2 T_f^3. \end{aligned}$$

Those associated to  $\gamma_1$  and  $\gamma_2$  correspond to monodromy around the toric hypersurface degeneration associated to the points  $D_1 \cap D_2$  and  $D_1 \cap D_4$ . The vanishing cycles for the twists  $T_i$  are given in figure 11.

One can calculate that  $E_A^s$  has precisely one cusp in the interior of  $\mathcal{X}_{\Sigma(A)}$ . This cusp yields the braid relations between  $T_4$  and  $T_i$  for  $i = 1, 2, 3$ . Adding these to the circuit relations, we obtain a finite presentation of  $\mathbf{G}_{\mathcal{C}_t}$ .

$$(113) \quad \mathbf{R} \rightarrow \langle T_1, \dots, T_4, \tilde{T}_1, \tilde{T}_2 \rangle \rightarrow \mathbf{G}_{\Sigma_t} \rightarrow 1$$

One can use this method for higher dimensions as well, but the understanding the singularities of  $E_A^s$  for  $(d+3)$ -sets is necessary to the completion of this project, as these generate necessary additional relations.

As a final remark, we observe that near  $\Sigma_\infty$ , we obtain a regeneration of the circuit  $C_4$ . Observing that  $D_4 \cdot E_A^s = 2$ , we see that the critical value in  $\Sigma_\infty$  splits into two values for each of the branches of the 2-fold étale cover, yielding a total of four critical values. To see the effect on the vanishing cycles, observe that the family  $\Sigma_{1/t}$  regenerates two extended circuits, each of which has a relation as given in section 4.3.3. This has the effect of gluing the degenerate vanishing cycles

together to obtain two vanishing cycles, for each branch of the étale cover, while parallel transport from one branch to the other yields a regenerated version of the involutions  $T_1 T_\infty$  on each regenerated circuit as given in section 4.3.3. However, to obtain the correct gluing formulas for these cycles requires a more nuanced control over the framing in the degenerate case.

## 5. APPLICATIONS

In this subsection we outline a strategy to decompose the directed Fukaya category associated to a pencil of toric hypersurfaces. After giving a combinatorial description of the decompositions, we discuss applications to the homological mirror symmetry conjecture for Fano toric stacks. While this conjecture has for the most part been settled, we show that our strategy may produce more detailed information on the structure of the equivalent categories. In particular, we will observe a finite collection of semi-orthogonal decompositions arising from edge paths in the secondary polytope. To each decomposition we formulate a conjectural homological mirror collection resulting from birational moves in the  $B$ -model setting.

**5.1. Landau-Ginzburg Degenerations.** We begin by returning to the original toric stack  $\mathcal{X}_Q$  associated to  $(Q, A)$  and the linear system  $\mathcal{L}_A \subset H^0(\mathcal{X}_Q, \mathcal{O}_A(1))$ . By the support of a section  $s \in \mathcal{L}_A \approx \mathbb{C}^A$ , we mean the set of non-zero coefficients. Given any subset  $A' \subset A$  and a section  $s = \sum_{\alpha \in A} c_\alpha e_\alpha \in \mathbb{C}^A$ , we say the restriction of  $s$  to  $A'$  is  $s|_{A'} = \sum_{\alpha \in A'} c_\alpha e_\alpha$ . By an  $A$ -pencil, we mean a pencil in  $\mathcal{L}_A$ . If it is clear from the context, we will simply write pencil for  $A$ -pencil. For what follows, we will consider  $A$ -pencils satisfying a strong, but common, property.

**Definition 5.1.** (i) Given  $A \subset \Lambda$  and  $A' \subset A$ , an  $A$ -pencil  $W \subset \mathbb{C}^A$  is  $A'$ -sharpened if it contains a full section and if  $s \in W$  implies  $s|_{A'} \in W$ .  
(ii) The Landau-Ginzburg, or LG model associated to an  $A'$ -sharpened pencil  $W$  is the induced map  $\mathbf{w} : \mathcal{X}_Q - D_W \rightarrow \mathbb{C}$  where  $D_W = \text{Zero}(s|_{A'})$  is the fiber over infinity of the pencil.

Our motivation to consider such pencils comes from homological mirror symmetry of Fano toric varieties ([33]). Given an  $d$ -dimensional Fano toric specified by a fan  $\Sigma$ , the Batyrev mirror is defined as  $\mathcal{X}_Q$  (or a crepant resolution thereof) with  $A$  equal to the union of 0 and the primitive generators of the one cones  $\Sigma(1)$ . A symplectic structure on the original variety then specifies a superpotential  $\mathbf{w}$  on  $(\mathbb{C}^*)^d \subset \mathcal{X}_Q$ . It turns out that  $\mathbf{w}$  is the LG model associated to a  $\{0\}$ -sharpened pencil  $W \subset \mathcal{L}_A$  on  $\mathcal{X}_Q$ . In fact, the case where  $A' = \{\alpha\}$  is a single element of  $A$  can simplify the discussion a bit because in such cases, any pencil containing  $e_\alpha$  is  $A'$ -sharpened. For now, though, we keep it general.

Given an  $A'$ -sharpened pencil we associate a sublattice  $\tilde{\Gamma}_{A'} \subset (\mathbb{Z}^A)^\vee$  generated by the cocharacter  $e_{A'}^\vee := \sum_{\alpha \in A'} e_\alpha^\vee$  and the subgroup  $\Gamma_{A'} \subset \Lambda_{A^\vee}$  equal to the image of  $\alpha_A^*(\tilde{\Gamma}_{A'})$ . We write  $f_{A'} = \alpha_A^\vee(e_{A'}^\vee)$ . These lattices give one parameter subgroups  $\tilde{G}_{A'}$  and  $G_{A'}$  in the tori acting on  $\mathcal{L}_A$  and  $\mathcal{X}_{\Sigma(A)}$  respectively. A useful fact concerning  $A'$  pencils is that they are stable under the  $G_{A'}$  action; more precisely, a pencil is  $A'$  sharpened if and only if it contains a full section and is stable under the action of  $G_{A'}$ .

The fan in  $\Gamma_{A'}$  with one cone generators  $\{\pm f_{A'}\}$  is  $\mathbb{P}^1$  modulo a finite cyclic group and the embedding  $\Gamma_{A'} \subset \Lambda_{A^\vee}$  gives a map from  $\mathbb{P}^1$  to  $\mathcal{X}_{\Sigma(A)}$  which we write

as

$$(114) \quad \phi : \mathbb{P}^1 \rightarrow \mathcal{X}_{\Sigma(A)}.$$

We would like to find a class of equivariant cycles equivalent to  $\phi(\mathbb{P}^1)$  in  $\mathcal{X}_{\Sigma(A)}$ . If we take a crepant resolution  $\tilde{\mathcal{X}}_{\Sigma(A)}$  of  $\mathcal{X}_{\Sigma(A)}$ , we may use ideas from Gromov-Witten theory of toric varieties. In particular, we may view  $\phi$  as an element of the space of stable maps  $\overline{\mathcal{M}}_{0,1}(\tilde{\mathcal{X}}_{\Sigma(A)}, [W])$ .

Taking the evaluation map  $ev : \overline{\mathcal{M}}_{0,1}(\tilde{\mathcal{X}}_{\Sigma(A)}, [W]) \rightarrow \tilde{\mathcal{X}}_{\Sigma(A)}$ , we define  $\mathcal{M}_W$  to be the fiber over the closure of the orbit containing  $\phi(\infty)$ . We make the following definition:

**Definition 5.2.** A fixed point  $\psi \in \mathcal{M}_W$  under the  $(\mathbb{C}^*)^A$  action will be called a maximal degeneration of  $W$ .

The first result we need is a combinatorial description of the maximal degenerations. For this, we review some terminology from [8], [47] and [9]. Let  $P \in \mathbb{R}^n$  be a  $n$  dimensional polytope with vertices  $\{p_1, \dots, p_m\}$  and  $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$  a linear map. We order the vertices so that if  $q_i := \gamma(p_i)$ , then  $q_i \leq q_j$  if  $i < j$  and write  $Q = \gamma(P)$ . Let  $\theta \in (\mathbb{R}^n)^\vee$  be linearly independent from  $\gamma$  and  $V_\theta$  the subspace spanned by  $\gamma$  and  $\theta$ . Assume that the half-plane  $H_\theta = \mathbb{R} \cdot \gamma \oplus \mathbb{R}_{>0} \cdot \theta$  intersects the normal fan of  $P$  transversely, i.e. every  $k$  dimensional cone in  $\mathcal{F}_\theta = V_\theta \cap N_P$  is the intersection of  $H_\theta$  with an  $(n-2+k)$  dimensional cone. Ordering the 2-cones  $\mathcal{F}_\theta(2) = \{\sigma_0, \dots, \sigma_r\}$  clockwise, one obtains the increasing sequence,  $p_{i_0} < \dots < p_{i_r}$  of points on  $P$  where  $p_{i_j}$  is the point dual to  $\sigma_j$ . From the construction, it is clear that  $\{p_{i_j}, p_{i_{j+1}}\}$  lie on an edge of  $P$  for any  $0 \leq j < r$ ,  $q_{i_0} = q_0$  and  $q_{i_r} = q_m$ . Any path  $\langle p_{i_0}, \dots, p_{i_r} \rangle$  obtained in this way is known as a parametric simplex path relative to  $\gamma$ .

In [8], these paths were realized as the vertices of the fiber polytope  $\Sigma_\gamma(P) := \Sigma(P, Q)$  called the monotone path polytope of  $P$ . Leaving a detailed review of fiber polytopes to the references above, we content ourselves to describe a theorem from [47]. Let  $G \approx (\mathbb{C}^*)^n$  be a complex torus acting on a projective toric variety  $X_\Sigma$  with fan  $\Sigma \subset G_\mathbb{R}^\vee$  where  $G^\vee = \text{Hom}(\mathbb{C}^*, G)$  and  $G^\wedge = \text{Hom}(G, \mathbb{C}^*)$  are the lattice of one parameter subgroups and characters respectively. Assume that  $L$  is an equivariant ample line bundle on  $X_\Sigma$  and  $P \subset G_\mathbb{R}^\wedge$  is the weight polytope for the action on  $H^0(X_\Sigma, L)$ . Elementary toric geometry gives  $\Sigma$  as the normal fan of  $P$ .

Suppose  $H \subset G$  is a subgroup and take  $E = H \cdot x$  for a non-boundary point  $x \in X_\Sigma$ . The Chow quotient  $X_\Sigma // H$  is defined as the closure of the orbit  $G \cdot E$  in the relative Chow variety of  $\dim(H)$  cycles of degree  $[E]$  in  $X_\Sigma$ . Write  $\pi_H : G_\mathbb{R}^\wedge \rightarrow H_\mathbb{R}^\wedge$  for the associated projection and take  $Q = \pi_H(P)$ . Then we state the following theorem which is an immediate corollary of results in loc. cit.:

**Theorem 5.3** ([47], [9]). *The Chow quotient  $X_\Sigma // H$  is a projective toric variety with  $G$  action and ample line bundle weight polytope equal to the fiber polytope  $\Sigma(P, Q)$ .*

Indeed, it was shown that  $\Sigma(P, Q)$  is the Newton polytope of the Chow form of  $E$ . We utilize this theorem to prove:

**Proposition 5.4.** *Suppose  $W$  is an  $A'$ -sharpened pencil. The maximal degenerations of  $W$  are in bijective correspondence with the vertices of the monotone path polytope  $\Sigma_{f_{A'}}(\Sigma(A))$ .*

The iterated fiber polytope  $\Sigma_{f_{A'}}(\Sigma(A))$  in this proposition was initially examined in [9].

*Proof.* We start by observing that the space  $\mathcal{M}_W$  admits a map to a Chow quotient of  $\mathcal{X}_{\Sigma(A)}$  as described above. Recall that  $\Gamma_W \subset \Lambda_{A^\vee}$  which gives a one parameter subgroup  $H \subset G := \Lambda_{A^\vee} \otimes \mathbb{C}^*$  of  $\Lambda_{A^\vee} \otimes \mathbb{C}^*$ , the torus acting on  $\mathcal{X}_{\Sigma(A)}$ . For any pointed pencil  $(W', s'_\infty)$  with  $s'_\infty = s_\infty$  and  $h \in H$ , one sees that  $h \cdot W' = W'$ . Furthermore, since the image of these under the projection to  $\mathcal{X}_{\Sigma(A)}$  are birationally equivalent to linear subspaces of  $\mathbb{P}(\Lambda_{A^\vee})$ , the collection of parameterized pointed pencils forms a dense open subset  $U$  of  $\mathcal{M}_W$ , with each such map being given  $g \cdot \phi$ . Let  $\mathcal{C}(\mathcal{X}_{\Sigma(A)}, [W])$  be the Chow variety of  $W$  in  $\mathcal{X}_{\Sigma(A)}$  and  $\tilde{\Phi} : \mathcal{M}_W \rightarrow \mathcal{C}(\mathcal{X}_{\Sigma(A)}, [W])$  the map associating a stable map to its pushforward cycle class. From the above discussion, the restriction of  $\tilde{\Phi}$  to  $U$  lands in the Chow quotient  $\mathcal{X}_{\Sigma(A)}/H$ , and as  $U$  is dense and the Chow quotient is closed, this induces the equivariant map  $\Phi : \mathcal{M}_W \rightarrow \mathcal{X}_{\Sigma(A)}/H$ . As this is a homeomorphism of  $U$  onto its image and equivariant, it extends to a homeomorphism on spaces and sends the fixed point set of  $\mathcal{M}_W$  to that of  $\mathcal{X}_{\Sigma(A)}$ . Applying theorem 5.3 gives that  $\mathcal{M}_W$  is equivariantly homeomorphic to the toric variety  $X_{\Sigma_{f_{A'}}(\Sigma(A))}$  associated to the monotone path polytope  $\Sigma_{f_{A'}}(\Sigma(A))$ . This confirms that the fixed points correspond bijectively to the vertices and proves the claim.  $\square$

A more detailed correspondence between the stack  $\mathcal{M}_W$  and the stack defined by  $\Sigma_{f_{A'}}(\Sigma(A))$  clearly exists by the proof above and the recent work [16]. For now we content ourselves with a study of the fixed points.

Given a maximal degeneration  $\psi \in \mathcal{M}_W$  associated to  $T = \langle t_{i_0}, \dots, t_{i_r} \rangle$ , we will write  $C_1, \dots, C_r$  for the components of the domain of  $\psi$  and  $c_1, \dots, c_r$  their images

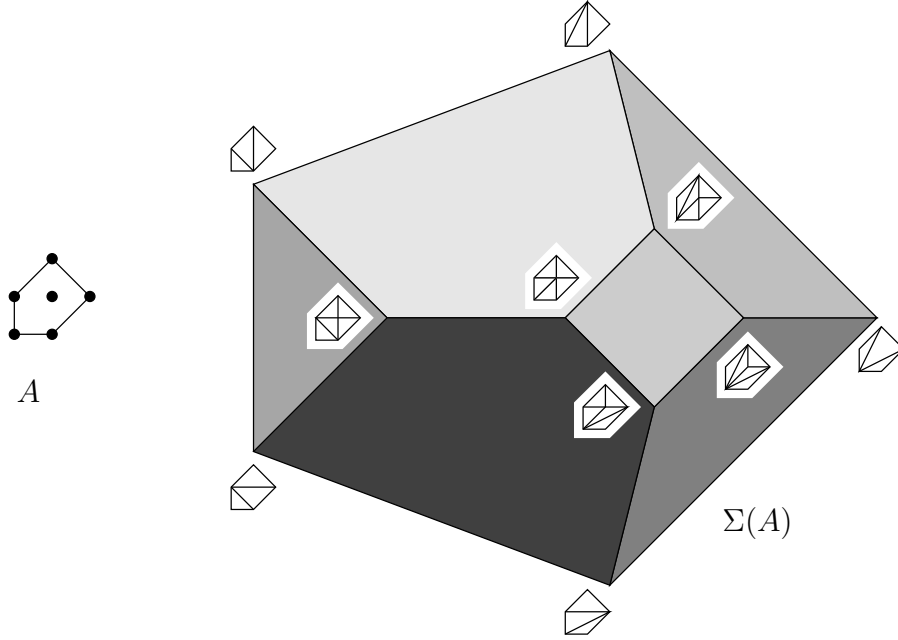


FIGURE 12.  $A$  and its secondary polytope

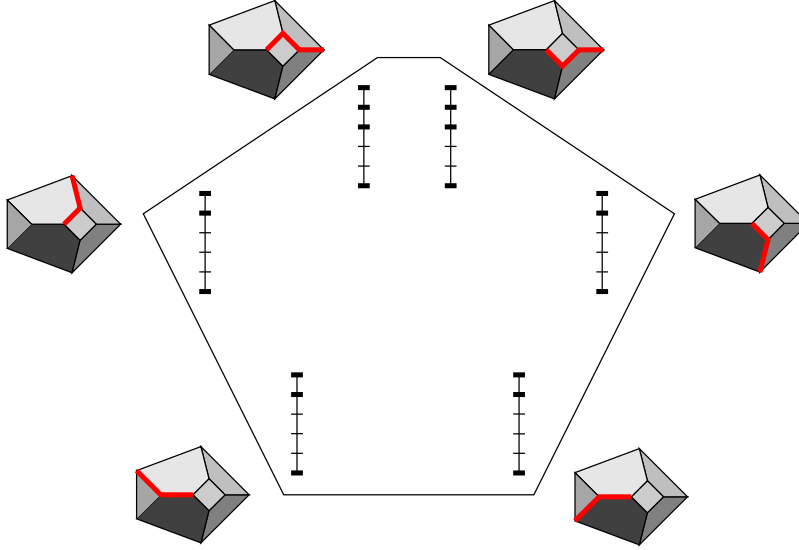


FIGURE 13. The monotone path polytope defined by the  $A'$ -sharpened pencil

in  $\mathcal{X}_{\Sigma(A)}$ . We will say that  $\psi$  has length  $r$  and for each  $1 \leq j \leq r$ , we will associate the pair of natural numbers  $(d_j, m_j)$  where  $d_j$  is the degree of  $\psi$  restricted to the  $j$ -th component, and  $m_j$  is the degree of  $E_A$  restricted to  $c_j$ . The total  $E_A^s$  degree of  $\psi$  is defined to be  $m_\psi = \sum_{j=1}^r m_j$ . Note that this yields the intersection degree of  $\psi$  with  $E_A^s$ . We call the data  $\mathbf{M}_\psi = (T, \{(d_j, m_j)\})$  a decorated simplex path and, for each such path, we define a decomposition of  $\mathbb{C}$  that we call a radar screen.

As we give this construction and other results, it will be useful to have an example to reference. We choose a sufficiently rich, but simple one arising as the homological, or Batyrev, mirror of  $\mathbb{P}^1 \times \mathbb{P}^1$  blown up at one point. More explicitly, we let  $A = \{(-1, 0), (0, -1), (0, 1), (1, 0), (-1, -1), (0, 0)\}$  and we consider  $A'$ -sharpened pencils where  $A' = \{(0, 0)\}$ . Recall from the comments above that such sharpened pencils consist of all pencils that contain  $e_{(0,0)} \in \mathbb{C}^A$  as a section. The secondary polytope is illustrated in figure 12 and the monotone path polytope is a (skew) hexagon which is represented in figure 13. Each vertex of the monotone path polytope is labelled with its coherent tight subdivision of the interval  $f_{A'}(\text{Sec}(A))$  inside the hexagon and the parametric simplex path on  $\Sigma(A)$  outside of the hexagon.

We now construct a decomposition of  $\mathbb{C}$  based on the information in  $\mathbf{M}_\psi = (T, \{(d_i, m_i)\})$  which we call a radar screen. To align the asymptotics correctly later, we define this decomposition in a fairly flexible fashion at first. Fix an increasing function  $g : T \rightarrow \mathbb{R} \cup \{\infty\}$  with  $g(i_0) = 0$  and  $g(i_r) = \infty$ . For any  $1 \leq j \leq r$  and any  $0 \leq k < m_j/d_j$  we take

$$(115) \quad C_{j,k} = \{z \in \mathbb{C} : g(i_j) \leq |z| < g(i_{j+1}), 2\pi k d_j / m_j \leq \arg(z) < 2\pi(k+1)d_j / m_j\}.$$

We totally order the collection  $\{C_{j,k}\}$  of regions so that  $C_{j,k} < C_{j',k'}$  if and only if  $j < j'$  or  $j = j'$  and  $k < k'$ . We now define a distinguished basis of paths  $\mathcal{B}_{\mathbf{M}_\psi} = \{\gamma_1, \dots, \gamma_{m_\psi}\}$  as in section 3.4 based at infinity and ordered so that if  $\gamma_l(1) \in C_{j,k}$  and  $\gamma_{l'}(1) \in C_{j',k'}$  with  $C_{j,k} < C_{j',k'}$  then  $l < l'$ . In order to make

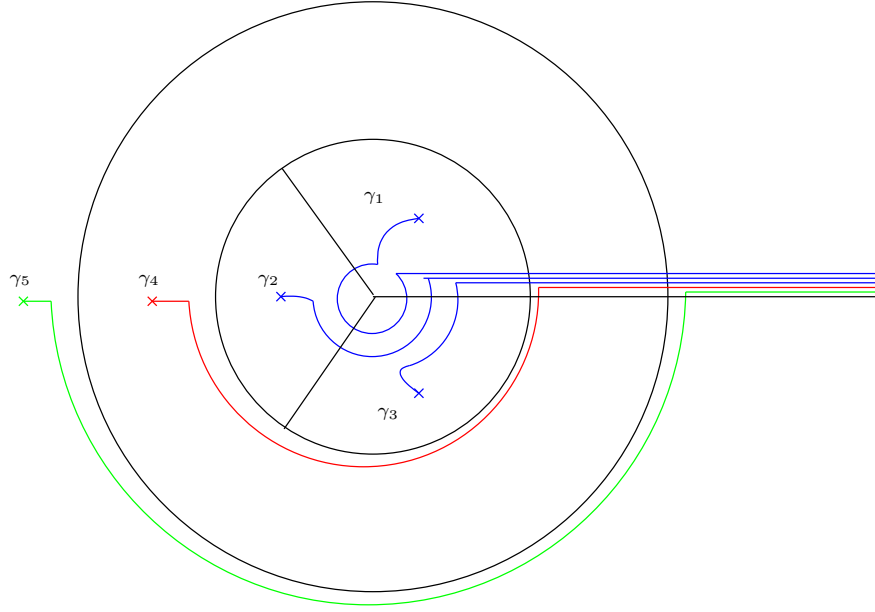


FIGURE 14. Radar screen for top vertices of monotone path polytope for  $A$

this collection precise, we fix a sufficiently small  $\varepsilon > 0$  and, for every  $j, k$  take  $s_{i_j} := m_{i_j}/d_{i_j}$  points  $\{p_1^{j,k}, \dots, p_{s_{i_j}}^{j,k}\}$  in  $C_{j,k}$  which are at least a distance  $2\varepsilon$  from the boundary of  $C_{j,k}$ . Let  $P = \cup_{j,k} \{p_1^{j,k}, \dots, p_{s_{i_j}}^{j,k}\}$  be the set of all such points.

For any  $1 \leq j \leq r$ ,  $0 \leq k < m_j/d_j$  and any  $\sum_{i=1}^{j-1} m_i + km_j/d_j < l \leq \sum_{i=1}^{j-1} m_i + (k+1)m_j/d_j$  we define the path  $\gamma'_l$  to be a horizontal line with  $\text{Im}(\gamma'_l) = \frac{(m-l)\varepsilon}{m}$  with  $\text{Re}(\gamma'_l(0)) = \infty$  and  $|\gamma'_l(1)| = g(j-1) + l\varepsilon/m$ . We take  $\gamma''_l : [0, 1-\varepsilon] \rightarrow \mathbb{C}$  to be a path with  $\gamma''_l(t) = e^{-2\pi i t d_j/m_j(m_j/d_j - k + 1/2)} \gamma'_l(1)$ . Let  $\tilde{\gamma}_l : [0, 1] \rightarrow \mathbb{P}^1$  be a rescaled concatenation of  $\gamma'_l$  with  $\gamma''_l$  and note that, for sufficiently small  $\varepsilon$ ,  $\tilde{\gamma}_l(1) \in C_{j,k}$ . We may then choose a set of  $l$  arbitrary non-intersecting paths  $\tilde{\gamma}'_l$  in  $C_{j,k}$  from  $\tilde{\gamma}_l(1)$  to  $p_n^{j,k}$  where  $n = l - (\sum_{i=1}^{j-1} m_i + km_j/d_j)$ . We then finally define  $\gamma_l$  to be the concatenation of  $\tilde{\gamma}$  with  $\tilde{\gamma}'$  to give a distinguished basis of paths from  $\infty$  to the set  $P$ .

To apply this construction, we examine a one parameter degeneration in  $\mathcal{M}_W$  to  $\psi$ . We need only choose a lattice point  $\theta \in (\mathbb{Z}^A)^\vee$  which is in the normal cone of the point in  $\Sigma_{f_{A'}}(\Sigma(A))$  corresponding to  $\psi$ . As above, this gives the fan  $\mathcal{F}_\theta$  and an embedding  $i : \mathcal{F}_\theta \rightarrow \mathcal{F}_{\Sigma(A)}$ . Letting  $\mathcal{X}_\theta$  be the stack associated to  $\mathcal{F}_\theta$ , we see that, quotienting  $\mathcal{F}_\theta$  by  $f_{A'}$ , we obtain a map  $F_{f_{A'}} : \mathcal{F}_\theta \rightarrow \mathbb{C}$  which is a toric degeneration of  $\mathbb{P}^1$ . It is clear that the zero fiber of  $F_{f_{A'}}$  is sent to  $\psi$  by  $i$  and that  $F_{f_{A'}}^{-1}(t)$  is an ordinary  $\mathbb{P}^1$  mapped to  $\mathcal{X}_{\Sigma(A)}$ .

Now,  $\psi$  corresponds to the monotone path  $T = \langle t_{i_0}, \dots, t_{i_r} \rangle$  on  $\Sigma(A)$ . Let  $s_j = f_{A'}(t_{i_j})$ , then  $f_{A'}(T)$  is a tight coherent subdivision of the marked interval  $[s_0, s_r]$ . Each subinterval  $[s_{j-1}, s_j]$  corresponds to an edge on  $\Sigma(A)$  and we may fill in all additional lattice points which are images of lattice points on  $\Sigma(A)$  to obtain



a modified sequence

$$(116) \quad \tilde{S} = \langle \tilde{s}_1, \dots, \tilde{s}_n \rangle$$

where  $\tilde{s}_j = f_{A'}(\tilde{t}_j)$ . We note here that it follows from [46] 10.1.G that  $m_j = s_j - s_{j-1}$  and  $d_j = m_j/e_j$  where  $e_j + 1$  is the number of lattice points on the edge  $\{t_{i_j}, t_{i_{j+1}}\}$ . It also follows that  $E_A$  pulls back via  $i : \mathcal{X}_\theta \rightarrow \mathcal{X}_{\Sigma(A)}$  to a section with non-trivial coefficients only on  $\{s_0, \dots, s_r\}$ . We define  $b_j = \theta(\tilde{t}_j)$  for every  $j$  and  $\mathbf{b} = (b_1, \dots, b_n)$ . Choosing another  $\theta$  if necessary, we may assume that  $b_0 = \dots = b_k = 0$  where  $\tilde{s}_k = s_1$ . Then  $\mathbf{b}$  defines the degeneration as in section 2.2 in that it defines a convex function on  $[s_0, s_r]$ .

Working on the level of coarse toric varieties as opposed to stacks, we may parameterize the degeneration using  $\mathbf{b}$  as follows. Identify  $V_\theta \cap (\mathbb{Z}^A)^\vee$  with  $(\mathbb{Z}^2)^\vee$  so that  $\mathcal{F}_\theta \subset (\mathbb{Z}^2)^\vee$  is dual to the upper convex hull of  $B_\theta = \{(\tilde{s}_j, b_j)\} \subset \mathbb{Z}^2$ . For toric varieties, we obtain a map  $\beta : \mathbb{C} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{P}^{n-1}$  given by

$$(117) \quad \beta(t, z) = (t, [t^{b_0} z^{\tilde{s}_0} : \dots : t^{b_n} z^{\tilde{s}_n}]).$$

The coarse variety  $X_\theta$  associated to  $\mathcal{X}_\theta$  is the closure of  $\text{im}(\beta)$  with coarse zero fiber  $\overline{F_\theta^{-1}(0)} := \overline{X_\theta(0)} = \cup_{j=1}^r C_j$ . Here  $C_j$  has moment polytope equal to the line segment from  $(\tilde{s}_{k_{j-1}}, b_{k_{j-1}})$  to  $(\tilde{s}_{k_j}, b_{k_j})$  where  $\tilde{s}_{k_j} = s_j$ . Let  $\mu_j = (b_{k_j} - b_{k_{j-1}})/(\tilde{s}_{k_j} - \tilde{s}_{k_{j-1}})$  be the slope of this line segment and define the map  $\alpha_j : \mathbb{R}_{\geq 0} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{C}^*$  via

$$(118) \quad \alpha_j(t, z) = (t, t^{-\mu_j} z).$$

Then we have the following proposition:

**Proposition 5.5.** *The map  $(\beta \circ \alpha_j)|_{\{t\} \times \mathbb{C}^*} : \mathbb{C}^* \rightarrow \overline{X}_\theta$  uniformly converges on compact sets to a  $d_j$  covering of  $C_j$  as  $t$  tends to 0.*

*Proof.* We simply compute

$$\begin{aligned} (\beta \circ \alpha_j)(t, z) &= (t, [t^{b_0} (t^{-\mu_j} z)^{\tilde{s}_0} : \dots : t^{b_n} (t^{-\mu_j} z)^{\tilde{s}_n}]), \\ &= (t, [t^{b_0 - \mu_j \tilde{s}_0} z^{\tilde{s}_0} : \dots : t^{b_n - \mu_j \tilde{s}_n} z^{\tilde{s}_n}]), \\ &= (t, [t^{(b_0 - b_{k_{j-1}}) - \mu_j(\tilde{s}_0 - \tilde{s}_{k_{j-1}})} z^{\tilde{s}_0 - \tilde{s}_{k_{j-1}}} : \dots \\ &\quad \dots : t^{(b_n - b_{k_{j-1}}) - \mu_j(\tilde{s}_n - \tilde{s}_{k_{j-1}})} z^{\tilde{s}_n - \tilde{s}_{k_{j-1}}}] ). \end{aligned}$$

By convexity, we have that the slope of the line segment connecting  $(\tilde{s}_i, b_i)$  to  $(\tilde{s}_{k_{j-1}}, b_{k_{j-1}})$  is strictly less than  $\mu_j$  for all  $i < k_{j-1}$  and strictly greater than  $\mu_j$  for all  $i > k_j$ . This implies that  $\kappa_i := (b_i - b_{k_{j-1}}) - \mu_j(\tilde{s}_i - \tilde{s}_{k_{j-1}}) \geq 0$  for all  $i$  with equality if and only if  $k_{j-1} \leq i \leq k_j$ . Utilizing this notation we have that

$$(119) \quad (\beta \circ \alpha_j)(t, z) = (t, [t^{\kappa_0} z^{\tilde{s}_0 - \tilde{s}_{k_{j-1}}} : \dots : 1 : \dots : z^{\tilde{s}_{k_j} - \tilde{s}_{k_{j-1}}} : \dots : t^{\kappa_n} z^{\tilde{s}_n - \tilde{s}_{k_{j-1}}}] ).$$

It is then clear that as  $t$  tends to 0,  $(\beta \circ \alpha_j)(t, z)$  converges pointwise to the map sending  $z$  to  $(0, [0 : \dots : 0 : 1 : \dots : z^{\tilde{s}_{k_j} - \tilde{s}_{k_{j-1}}} : 0 : \dots : 0])$  which is a degree  $d_j$  cover of  $C_j$ . Uniform convergence on compact sets then follows.  $\square$

We utilize this in the proof of the following theorem:

**Theorem 5.6.** *Let  $\psi$  be a maximal degeneration of a LG model associated to  $A$ . If  $\psi_t \in \mathcal{M}_W$  is sufficiently close to  $\psi$ , there exists a radar screen  $\mathbf{M}_\psi$  decomposition of the domain of  $\psi_t$  such that the paths of the distinguished basis  $\{\gamma_1, \dots, \gamma_m\}$  end on the critical values of the LG model associated to  $\psi_t$ .*

*Proof.* For any  $\epsilon$  let  $\mathbb{P}^1(\epsilon)$  consist of all points in  $\mathbb{P}^1$  that are at least  $\epsilon$  distance from 0 and  $\infty$ . From [46] 10.1, we have that  $\{E_A^s = 0\} \cap C_j$  consists of a single point  $\{q_j\}$  for every  $j$ . It then follows from proposition 5.5 that for any small  $\epsilon$  and  $\kappa > 0$ , there exists  $\delta > 0$  so that for  $t < \delta$  and every  $1 \leq j \leq n$  the function  $(\beta \circ \alpha_j)|_{\{t\} \times \mathbb{P}^1(\kappa)}$  is  $\epsilon$  close to the  $d_j$  covering  $(\beta \circ \alpha_j)|_{\{0\} \times \mathbb{P}^1(\kappa)}$ . In particular, from the comment above, we may choose  $\epsilon$  and  $\kappa$  small enough so that

$$(120) \quad \{E_A^s = 0\} \cap \beta(t, \mathbb{C}^*) = \{E_A^s = 0\} \cap \left( \bigcup_{j=1}^n (\beta \circ \alpha_j)|_{\{t\} \times \mathbb{P}^1(\kappa)} \right)$$

for  $t < \delta$ . Let  $\mathcal{C}_{t,j}(\kappa) = (\beta \circ \alpha_j)|_{\{t\} \times \mathbb{P}^1(\kappa)}$  and  $\mathcal{C}_t(\kappa) = \bigcup_{j=1}^n (\beta \circ \alpha_j)|_{\{t\} \times \mathbb{P}^1(\kappa)}$ , then it is clear that we may choose  $\epsilon$  sufficiently small so that every component in the union is disjoint. Fix such a  $\epsilon$  and  $\kappa$  so that the above equation holds and, let  $N = \max\{\frac{2}{\mu_{j+1} - \mu_j}\}$  and  $\delta_0 = \min\{\delta^N, \kappa\}$ . Then a straightforward argument shows that there exists a sequence  $\{g_2, \dots, g_{r-1}\}$  such that, for all  $t < \delta_0$ ,

$$(121) \quad \frac{1}{\delta_0} t^{-\mu_i-1} \leq g_i \leq \delta_0 t^{-\mu_i}.$$

We define  $g_{\epsilon, \kappa} : T \rightarrow \mathbb{R}$  via  $g_{\epsilon, \kappa}(t_{i_0}) = 0$ ,  $g_{\epsilon, \kappa}(t_{i_r})$  and  $g_{\epsilon, \kappa}(t_{i_j}) = g_j$ . We observe that for  $0 < t < \delta_0$  and any  $z \in \mathcal{C}_t(\kappa)$  we have that  $z = \mathcal{C}_{t,j}(\kappa)$  if and only if  $z = t^{-\mu_j} w$  for some  $\delta_0 < |w| < 1/\delta_0$ . This implies that  $z \in \mathcal{C}_{t,j}(\kappa)$  is contained in this image only if  $\delta_0 t^{-\mu_j} < |z| < t^{-\mu_j}/\delta_0$ . Thus for  $z \in \mathcal{C}_t(\kappa)$  we have  $z \in \mathcal{C}_{t,j}(\kappa)$  if and only if  $g_{\epsilon, \kappa}(t_{i_j}) < |z| < g_{\epsilon, \kappa}(t_{i_{j+1}})$ . By equation 120 and proposition 5.5, this implies that the points  $\{E_A^s = 0\} \cap \mathcal{C}_{t,j}(\kappa)$  are, after a rotation, contained in the interior of the components  $C_{j,k}$  for  $0 \leq k \leq d_j$  of the radar screen for  $\mathbf{M}_\psi$  with radial function  $g_{\epsilon, \kappa}$ . Indeed, because we may choose  $\epsilon$  small enough that  $\mathcal{C}_{t,j}(\kappa)$  is approximately a  $d_j$ -fold covering of  $C_j$ , we have that the  $2\pi/d_j$  angular regions each approximately cover  $C_j$  once and the intersection of  $E_A^s = 0$  with each such map contains  $m_j/d_j$  points (the order of  $E_A^s$  restricted  $C_j$ ), justifying that this radar screen is associated  $\mathbf{M}_\psi = (T, \{(d_i, m_i)\})$ . By definition, the degenerate values of the LG model  $\psi_t$  are the intersection points of  $\beta(t, \cdot)$  with  $E_A^s = 0$  and, again, by equation 120, all such points are accounted for in the interiors of the regions  $\mathcal{C}_{t,j}(\kappa)$ .  $\square$

To each annular region in a radar screen, we can define a partial LG model by regenerating the associated circuit. In fact, the proof of the proposition above gives precise control on the simultaneous regeneration of every circuit in the maximal degeneration. We recall that a partial LG model is simply a LG model defined over a curve  $\Sigma$ , such as an annulus, rather than a disc or  $\mathbb{C}$ , along with some conditions on the fiber ([40]). We will neglect these conditions and simply require that this is a framed Lefschetz pencil. One may rigorously construct examples of such partial models as the pullbacks of  $\mathcal{X}_{\Theta(A)}$  via the map  $(\beta \circ \alpha_j)|_{\{t\} \times \mathbb{P}^1(\kappa)}$  as above. Utilizing theorem 5.6, for every maximal degeneration of a LG model  $\psi$ , we obtain a semi-orthogonal decomposition of a category which can be thought of as a type of directed Fukaya category (see [11], [55]). As this category has not yet been defined in general, we will examine the special case for which an  $A'$ -sharpened pencil gives rise to the directed Fukaya category of a Lefschetz pencil as defined in [24].

**Proposition 5.7.** *Let  $A' \subset \text{Int}(Q)$  and  $W$  be a generic  $A'$ -sharpened pencil. Then the LG model  $\mathbf{w}$  associated to  $W$  has isolated Morse critical points away from  $\infty$ .*

*Proof.* Observe first that the intersection of  $W$  with the components  $\Delta_{Q_i} = 0$  of  $E_A^s = 0$  corresponding to proper faces  $Q_i$  of  $Q$  is trivial. These intersections give stratified Morse critical points, so a trivial intersection implies that the only intersection of  $W$  with  $E_A^s = 0$  is that with the full discriminant and the boundary. The discriminant intersection points correspond to Morse critical points of the LG model and the boundary intersection points, or toric hypersurface degeneration points, occur only at 0 and  $\infty$ . Since the generic model associated to an  $A'$ -sharpened pencil sends 0 to a point on the interior of the face of  $\Sigma(A)$  corresponding to the subdivision  $S = \{(Q, A - A')\}$ , the fiber over zero is smooth.

To check that the intersection of  $W$  with  $\Delta_{Q_i} = 0$  is trivial, we may assume that  $W$  is near a maximal degeneration point  $\psi$ . Recall that the sequence  $c_1, \dots, c_r$  referred to the sequence of equivariant lines on  $\mathcal{X}_{\Sigma(A)}$  in the image of  $\psi$ . By theorem 5.6, the  $W$  intersects a component  $\Delta_{Q_i} = 0$  of  $E_A^s = 0$  if and only if  $c_j$  intersects  $\Delta_{Q_i} = 0$  for some  $j$ . For each  $c_j$  there is a circuit  $B_j \subset A$  for which  $E_A^s|_{c_j} = (E_{B_j}^s)^{m_j/d_j}$ . By the product formula of [46], we have that  $\{\Delta_{Q_i} = 0\} \cap c_j \neq \emptyset$  if and only if  $B_j \subset Q_i$ . Assume that this is the case and let  $T_{\pm}$  be the two triangulations corresponding to  $c_j$  with points  $\phi_{\pm} \in \Sigma(A)$  such that  $f_{A'}(\phi_+) > f_{A'}(\phi_-)$ . Then, by the formula for  $\phi_{\pm}$  and the definition of  $f_{A'}$ , we have that

$$(122) \quad f_{A'}(\phi_{\pm}) = \sum_{\alpha \in A'} \left( \sum_{\alpha \in \sigma, \sigma \in T_{\pm}} \text{Vol}(\sigma) \right).$$

Let  $\tilde{B}_j$  be any extended circuit in  $A$  containing  $B_j$  such that  $c_j$  modifies triangulations associated to  $\tilde{B}_j$ , i.e.  $T_{\pm}$  restricts to the two triangulations  $T_{\pm}^j$  on  $\text{Conv}(\tilde{B}_j)$ . If  $\alpha \in \tilde{B}_j \cap A'$  then  $\alpha \notin B_j$  since  $A' \cap \partial Q = \emptyset$  and  $B_j \subset Q_j \subset \partial Q$ . But then by equation 83, for every such  $\alpha$ , the sum of the volumes is given by

$$(123) \quad \text{Vol}(\text{Conv}(\tilde{B}_j)) = \sum_{\alpha \in \sigma, \sigma \in T_+^j} \text{Vol}(\sigma) = \sum_{\alpha \in \sigma, \sigma \in T_-^j} \text{Vol}(\sigma),$$

contradicting  $f_{A'}(\phi_+) > f_{A'}(\phi_-)$ . Thus  $\{\Delta_{Q_i} = 0\} \cap c_j = \emptyset$  for all  $1 \leq j \leq r$  which was to be shown.  $\square$

As was mentioned above, given a LG model  $\mathbf{w}$  with Morse singularities and reasonable boundary conditions, i.e. a symplectic Lefschetz pencil, one may define a directed Fukaya category as in [55]. Taking the paths  $\mathcal{B}$  associated to a radar screen to be the generating exceptional collection, theorem 5.6 and the above proposition give:

**Corollary 5.8.** *For every maximal degeneration of a LG model associated to  $A$ , there exists a smooth LG model  $\psi_t$  and a semi-orthogonal decomposition of the directed Fukaya category:*

$$(124) \quad \text{Fuk}^{\rightarrow}(\psi_t) \approx \langle \mathcal{T}_1, \dots, \mathcal{T}_r \rangle$$

where  $\mathcal{T}_i$  is the directed Fukaya category of a regenerated circuit corresponding to  $\psi|_{C_i}$ .

**5.2. Homological mirror symmetry.** In the final pages of this article, we will detail a conjectural homological mirror to the maximally degenerated LG model and present some supporting evidence for this viewpoint. Aside from the intrinsic interest which many have for the subject of homological mirror symmetry, the

perspective obtained from maximal degenerations predicts many results in the  $B$ -model setting which previously are either unknown or have been approached from a more opaque angle.

We restrict our conversation to the homological mirrors of nef Fano DM toric stacks, by which we mean we choose a simplicial stacky fan  $\Sigma = (\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}^d, \beta, \Sigma)$  with one cone generators  $\Sigma(1) \subset \mathbb{Z}^d$  such that  $\Sigma(1) \subset \partial(\text{Conv}(\Sigma(1)))$ . This condition is equivalent to  $-K_{\mathcal{X}_\Sigma}$  being nef. Letting  $\alpha_0 = 0 \in \mathbb{Z}^d$ , we define the  $A$ -model mirror of  $\mathcal{X}_\Sigma$  to be a generic LG model  $\mathbf{w}$  associated to a  $\{\alpha_0\}$ -sharpened pencil  $W$  for the set  $A = \Sigma(1) \cup \{\alpha_0\}$ . It is not hard to show that any homological mirror of a toric Fano orbifold as defined by [33] can be obtained in this way. We now formulate the structure on  $\mathcal{X}_\Sigma$  corresponding to a maximal degeneration  $\psi$  of  $\mathbf{w}$ .

For any triangulation  $T$  of  $A$ , we define a stacky fan  $\Sigma_T$  as follows. Let  $\sigma \in T$  be a simplex which contains  $\alpha_0$  and  $\tau$  be the minimal face of  $\sigma$  containing  $\alpha_0$  and  $\tau(1)$  the vertices of  $\tau$ . We write  $\Lambda_T$  for the finite rank abelian group  $\mathbb{Z}^d / \text{Lin}_{\mathbb{Z}}(\tau(1))$  and  $\lambda : \mathbb{Z}^d \rightarrow \Lambda_T$  the quotient. The star  $St_T(\tau)$  of  $\tau$  in  $T$  is defined to be the collection of simplices in  $T$  containing  $\tau$  as a face. For each such simplex  $\sigma \in St_T$  we define the cone  $S_\sigma = \text{Cone}(\{\lambda(v) : v \in \sigma(1)\}) \subset \Lambda_T \otimes \mathbb{R}$  with generators  $\lambda(v) \in \Lambda_T$ . The collection of cones  $\{S_\sigma\}$  along with their intersections defines a stacky fan which we write as  $\Sigma_T$ .

**Definition 5.9.** Let  $\psi$  be a maximal degeneration with decorated simplex path  $\mathbf{M}_\psi = (T, \{(d_i, m_i)\})$  where  $T = \langle t_0, \dots, t_{r+1} \rangle$ . The sequence of stacks

$$(125) \quad \mathbf{S}_\psi = (\mathcal{X}_{\Sigma_{t_{r+1}}}, \dots, \mathcal{X}_{\Sigma_{t_0}})$$

will be called the mirror sequence to the  $\psi$ .

Let us write out an example of the mirror sequences for the maximal degenerations of  $\{\alpha_0\}$ -pencils on the variety  $\mathcal{X}_Q$  of the previous section. Referring to figure 13, we enumerate the maximal degenerations  $\psi_1, \psi_2$  and  $\psi_3$  associated to the vertices on the right hand side of the monotone path polytope, starting from the top and ending on the bottom. The mirror sequences of these degenerations are

$$\begin{aligned} \mathbf{S}_{\psi_1} &= (\mathcal{X}_Q^{mir}, F_1, \mathbb{P}^2, \{pt\}), \\ \mathbf{S}_{\psi_2} &= (\mathcal{X}_Q^{mir}, F_1, \mathbb{P}^1), \\ \mathbf{S}_{\psi_3} &= (\mathcal{X}_Q^{mir}, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1). \end{aligned}$$

Noting that  $F_1$  is the projective line bundle of  $\mathcal{O}(-1) \oplus \mathcal{O}$  over  $\mathbb{P}^1$  for the second sequence and that  $\mathbb{P}^1 \times \mathbb{P}^1$  is the trivial projective line bundle over  $\mathbb{P}^1$  for the third, this example suggests that the mirror sequences to maximal degenerations correspond to runs of the minimal model program for the mirror.

We briefly recall the minimal model program on toric varieties as presented in [48], [39] and [38]. For the moment, we consider  $\Sigma$  to be a general complete, simplicial stacky fan in  $\mathbb{Z}^d$  which is not necessarily nef Fano. If  $\alpha_i \in \Sigma(1)$ , write  $r_i$  for the index of  $\mathbb{Z} \cdot \alpha_i$  in  $\mathbb{R} \cdot \alpha_i \cap \mathbb{Z}^d$  and  $X_\Sigma$  for the toric variety with quotient singularities. Consider the stack  $\mathcal{X}_\Sigma$  to be a log crepant resolution of the toric variety  $X_\Sigma$  with  $\mathbb{Q}$ -divisor  $B = \sum_{\alpha_i \in \Sigma(1)} \frac{r_i - 1}{r_i} D_i$  where  $D_i$  is the divisor corresponding to the one cone  $\mathbb{R} \cdot \alpha_i$ .

Given a codimension 1 cone  $w = \langle \alpha_3, \dots, \alpha_{d+1} \rangle$  corresponding to an extremal rational curve, there exist precisely two maximal cones containing  $w$  with additional

vertices  $\alpha_{i_1}$  and  $\alpha_{i_2}$ . The set  $C(w) = \{\alpha_0, \alpha_1, \dots, \alpha_{d+1}\}$  is an extended circuit and has a fundamental relation

$$(126) \quad \sum_{j=0}^{d+1} \tilde{a}_j \alpha_j = 0,$$

$$(127) \quad \sum_{i=0}^{d+1} \tilde{a}_i = 0,$$

as in equation 78. We assume  $(\tilde{a}_1, \dots, \tilde{a}_{d+1}) = 1$  and that  $\Lambda_w \subset \mathbb{Z}^d$  is the lattice generated by  $\{\alpha_j : 1 \leq j \leq d+1\}$ . As was noted in section 4.1, the volume  $\text{Vol}_0(C(w)) := \text{Conv}(\{\alpha_1, \dots, \alpha_{d+1}\})$  is given by  $i_w \cdot \sum_{i=1}^{d+1} \tilde{a}_i$  where  $i_w$  is the index  $[\mathbb{Z}^d : \Lambda_w]$ . This volume will become a relevant invariant later in the section. As we will reference [39] heavily, we note that there the primitive vectors  $v_i$  are  $\alpha_i/r_i$  and that  $a_i = r_i \tilde{a}_i / r_w$  where  $r_w = \gcd(r_1 \tilde{a}_1, \dots, r_{d+1} \tilde{a}_{d+1})$  so that,  $\gcd(a_1, \dots, a_{d+1}) = 1$ . We orient the circuit so that  $\tilde{a}_0 < 0$  (i.e.  $\alpha_0 \in C_-(w)$ ) and take the signature of  $C(w)$  to be  $\sigma(C(w)) = (p, q; r)$ .

For any  $\alpha_j \in C_+(w)$ , let  $\tau_j = \text{Cone}(\{\alpha_i \in \text{Core}(C(w)) : i \neq j\})$ . We state a proposition which essentially rephrases [48] 14-2-1.

**Proposition 5.10** ([48], 14-2-1). *The fan consisting of  $\Upsilon = \{\tau_j : \alpha_j \in C_+(w)\}$  is contained in  $\Sigma$ . Moreover, there exists a collection of  $r$ -cones  $\text{Supp}(w) := \{\sigma_i : 1 \leq i \leq m\}$  in  $\Sigma$  such that the star of  $\Upsilon$  consists of cones  $\tilde{\Upsilon} := \{\tau_j + \sigma_i : \alpha_j \in C_+(w), 1 \leq i \leq m\}$ .*

This proposition allows us to relate the fan structure around the extremal rational curve given by  $w$  with the circuit  $\text{Core}(C(w))$ . To make this precise, we define the collection of simplices  $\text{Simp}(\Sigma) = \{\text{Conv}(\sigma(1) \cup \{\alpha_0\}) : \sigma \in \Sigma(d)\}$  and write  $\text{Vol}(\Sigma) = \sum_{\sigma \in \text{Simp}(\Sigma)} \text{Vol}(\sigma)$ . Note that if  $X_\Sigma$  is projective, then  $\text{Simp}(\Sigma)$  extends to a regular triangulation  $T$  of  $\text{Conv}(A)$ . Indeed, taking a very ample divisor on  $X_\Sigma$  and its graph  $\psi_D$  on  $\mathbb{R}^d$  as defined in [26], section 3.4, admits a perturbation as a function in  $\mathbb{R}^{\Sigma(1)}$  which yields regular triangulation. We call  $T$  a convex extension of  $\text{Simp}(\Sigma)$ .

**Corollary 5.11.** *Let  $T$  be a convex extension of  $\text{Simp}(\Sigma)$  and  $w \in \Sigma(n-1)$  an extremal rational curve. Then  $T$  is supported on  $\text{Core}(C(w))$ .*

This corollary ought to be thought of as part of the correspondence between the secondary fan and the secondary polytope as given in [19]. Let us review the extremal contraction associated to  $w$ , the structure of which can be phrased simply given the corollary:

**Proposition 5.12.** *Let  $\Sigma$  be a stacky fan and  $w$  an extremal curve in  $\mathcal{X}_\Sigma$ . Let  $T$  be a convex extension of  $\text{Simp}(\Sigma)$  and  $s_{C(w)}(T)$  the circuit modification of  $T$  along  $C(w)$  and  $\Sigma' = \Sigma_{s_{C(w)}(T)}$ . The extremal contraction of  $w$  is given by a birational map*

$$(128) \quad f_w : \mathcal{X}_\Sigma \dashrightarrow \mathcal{X}_{\Sigma'}$$

To be more precise about the how the circuit modification changes the fan structure, we may define the collection of cones  $\Upsilon^- = \{\text{Cone}(C(w) - \alpha_j) : \alpha_j \in C_-(w)\}$  and the collection of cones  $\bar{\Upsilon}^- := \{\tau + \sigma : \tau \in \Upsilon^-, \sigma \in \text{Supp}(w)\}$ . Then replacing the cones of  $\Sigma$  in  $\bar{\Upsilon}$  with those in  $\bar{\Upsilon}^-$  yields the fan  $\Sigma'$ .

We note that, by a birational map of toric stacks we mean a log map of underlying log varieties as in [38]. While we refer to the above references for the proof of this proposition, we will detail the three essentially different situations that can occur. They may be distinguished by the signature  $\sigma(C(w)) = (p, q; r)$  of  $C(w)$ . For this, we need to define three stacks associated to  $w$ . Define the lattices

$$\begin{aligned}\Lambda_F &= \frac{\text{Lin}_{\mathbb{R}}(\text{Core}(w)) \cap \mathbb{Z}^d}{\text{Lin}_{\mathbb{R}}(C_-(w)) \cap \mathbb{Z}^d}, \\ \Lambda_E &= \frac{\mathbb{Z}^d}{\text{Lin}_{\mathbb{R}}(C_-(w)) \cap \mathbb{Z}^d}, \\ \Lambda_B &= \frac{\mathbb{Z}^d}{\text{Lin}_{\mathbb{R}}(\text{Core}(w)) \cap \mathbb{Z}^d}\end{aligned}$$

with projections  $\pi_E : \mathbb{Z}^d \rightarrow \Lambda_E$ ,  $\pi_B : \mathbb{Z}^d \rightarrow \Lambda_B$  and  $\pi_F : \text{Lin}_{\mathbb{R}}(\text{Core}(w)) \cap \mathbb{Z}^d \rightarrow \Lambda_F$ . We define stacky fans  $\Sigma_F = \{\pi_F(\tau) : \tau \in \Upsilon\}$ ,  $\Sigma_E = \{\pi(\sigma \cup \tau) : \sigma \in \text{Supp}(w), \tau \in \Upsilon\}$  and  $\Sigma_B = \{\pi_B(\sigma) : \sigma \in \text{Supp}(w)\}$ ; write their associated stacks as  $\mathcal{F}$ ,  $\mathcal{E}$  and  $\mathcal{B}$  with coarse spaces  $F$ ,  $E$  and  $B$ . Then it is not hard to see that proposition 5.10 implies that the induced map  $\pi : \mathcal{E} \rightarrow \mathcal{B}$  is a smooth fibration with fiber  $\mathcal{F}$ . We also write the a natural inclusion of  $\mathcal{E}$  in  $\mathcal{X}_{\Sigma}$  as  $j : \mathcal{E} \rightarrow \mathcal{X}_{\Sigma}$ .

To simplify, we order  $C(w) = \{\alpha_0, \dots, \alpha_{d+1}\}$  so that  $C_+(w) = \{\alpha_i : 1 \leq i \leq p\}$ ,  $C_0(w) = \{\alpha_{p+1}, \dots, \alpha_{p+r}\}$  and  $C_-(w) = \{\alpha_0, \alpha_{p+r+1}, \dots, \alpha_{p+q+r-1}\}$ . We also fix  $T$  to be a convex extension of  $\text{Simp}(\Sigma)$ ,  $s_{C(w)}(T)$  the circuit modification of  $T$  by  $C(w)$ , and write  $\Sigma'$  for the stacky fan  $\Sigma_{s_{C(w)}(T)}$ .

We start with the case of  $q = 1$ . Here we have that  $\mathcal{E} = \mathcal{X}_{\Sigma}$  and that the stack  $\mathcal{F}$  is covered by a weighted projective space  $\mathbb{P}(a_1, \dots, a_p)$ . In this case the map  $\pi_B : \mathbb{Z}^d \rightarrow \Lambda_B$  induces a map of stacky fans from  $\Sigma = \Sigma_E$  onto  $\Sigma_B$  which gives the smooth map  $f : \mathcal{X}_{\Sigma} \rightarrow \mathcal{B}$ . It is clear then that  $\Sigma_B$  equals  $\Sigma'$  and that  $f = \pi$  is a Mori fiber space map with fiber equal to  $\mathcal{F}$ .

Now let us examine the case of  $q = 2$ . In this case  $C_-(w) = \{\alpha_0, \alpha_{d+1}\}$  so that  $\Lambda_E$  has rank  $(d - 1)$  and  $\mathcal{E}$  is a divisor in  $\mathcal{X}_{\Sigma}$ . It is not hard to see in this case that  $\Sigma'$  is obtained by replacing the cones in the star of  $\Upsilon$  with the stacky cones  $\{\sigma \cup C_+(w) : \sigma \in \text{Supp}(w)\}$ . In other words, we delete the one cone corresponding to  $\alpha_{d+1}$  which, on the coarse level, gives a divisorial contraction  $f : X_{\Sigma} \rightarrow X_{\Sigma'}$  whose exceptional locus is  $E$  blown up along  $B$ . As was mentioned in [38], there is not always a morphism  $f : \mathcal{X}_{\Sigma} \rightarrow \mathcal{X}_{\Sigma'}$  reducing to this map, but it is not difficult to find an étale correspondence which induces the birational map.

The case of  $q > 2$  corresponds to an equivariant flip. Indeed, as above, we may take the stacky fan  $\tilde{\Sigma}$  by replacing the star of  $\Upsilon$  by all stacky cones  $\{\sigma \cup C_+(w) : \sigma \in \text{Supp}(w)\}$ . Then the induced map  $\tilde{\pi} : \mathcal{X}_{\Sigma} \rightarrow \mathcal{X}_{\tilde{\Sigma}}$  contracts  $\mathcal{E}$  which in this case has codimension  $> 1$  and contains the rational curve corresponding to  $w$ . As  $\mathcal{X}_{\tilde{\Sigma}}$  is singular, to obtain the flip  $\phi : \mathcal{X}_{\Sigma'} \rightarrow \mathcal{X}_{\tilde{\Sigma}}$  one need only observe that  $K_{\mathcal{X}_{\Sigma'}}$  is ample relative to  $\phi$ .

With these cases in mind, we define a minimal model program, or MMP, sequence as follows.

**Definition 5.13.** Given a toric stack  $\mathcal{X} = \mathcal{X}_r$ , a sequence of equivariant birational maps

$$(129) \quad \mathcal{X}_r \xrightarrow{f_r} \mathcal{X}_{r-1} \dashrightarrow \dots \dashrightarrow \mathcal{X}_0$$

will be called a MMP sequence of  $\mathcal{X}$  if for every  $1 \leq i \leq r-1$ ,  $f_i$  is a divisorial contraction or flip and  $f_r$  is a Mori fiber space.

With this definition in hand, we may now state a suggestive theorem relating maximal degenerations of LG models to the minimal model program.

**Theorem 5.14.** *The set of MMP sequences of  $\{\mathcal{X}_\Sigma : A = \Sigma(1) \cup \{0\}, \mathcal{X} \text{ is nef Fano}\}$  are in bijective correspondence with the set of mirror sequences to maximal degenerations of  $\{\alpha_0\}$ -sharpened pencils on  $\mathcal{X}_Q$ . Both are in bijective correspondence with vertices of the monotone path polytope  $\Sigma_{\rho_{\alpha_0}}(\Sigma(A))$ .*

*Proof.* We observe that  $\rho_{\alpha_0} : \Sigma(A) \rightarrow [0, \text{Vol}(Q)] \subset \mathbb{R}$ . By proposition 5.4 we have that the vertices of the monotone path polytope are in bijective correspondence with maximal degenerations. For any such vertex  $\psi$ , let  $\mathbf{M}_\psi = (c_i, d_i, m_i, t_i)$  by its degeneration data. We first observe that the mirror sequence to  $\psi$  is a MMP sequence for  $\mathcal{X}_\Sigma$ . Taking  $\rho_{\alpha_0}(t_{r+1}) = \text{Vol}(Q)$  to have the maximal value, we must have that  $\Sigma_{t_{r+1}}$  is nef Fano. For every circuit  $c_i$ , we have that  $\alpha_0 \notin (c_i)_+$  which implies that there is an extremal contraction  $f_i : \mathcal{X}_{\Sigma_{t_{i+1}}} \dashrightarrow \mathcal{X}_{\Sigma_{t_i}}$  corresponding to the circuit. If  $1 \leq i \leq r$  then since  $\rho_{\alpha_0}(t_i) \neq 0$ , we have that  $\alpha_0$  is a vertex of the a simplex in  $t_i$ . This implies that  $\sigma(c_i) = (p, q; r)$  with  $q \geq 1$  so that  $f_i$  is a divisorial contraction or a flip. On the other hand, if  $i = 0$ , then  $\rho_{\alpha_0}(t_0) = 0$  which implies that  $\alpha_0$  is not a vertex of any simplex of  $t_0$ . This implies that  $\sigma(c_0) = (p, 1; r)$  and that  $f_0$  is a Mori fiber space. Therefor the mirror sequence to  $\psi$  corresponds to a MMP sequence for  $\mathcal{X}_\Sigma$ . The converse is obtained by running the above correspondences in reverse.  $\square$

From this result, one is led to conjecture that every decomposition of the  $A$ -model given by a radar screen corresponds to an equivalent decomposition of the  $B$ -model associated to the MMP sequence and the derived category of  $\mathcal{X}_\Sigma$ . On the  $B$ -model side, such a decomposition has been given very explicitly by Kawamata in [39]. We write a condensed version of his results here, with a short proof translating his notation to ours.

**Theorem 5.15** ([39]). (i) *Let  $C(w) = \{\alpha_0, \dots, \alpha_{d+1}\}$  correspond to a signature  $(p, q; r)$  circuit in a rank  $\mathbb{Z}^d$  lattice with  $\alpha_0 = 0$  and triangulations  $T_\pm$ . Letting  $\mathcal{X} = \mathcal{X}_{\Sigma_{T_+}}$ , the derived category  $D^b(\mathcal{X})$  has a strong exceptional collection of  $\text{Vol}_0(A)$  line bundles*

$$(130) \quad \mathbf{E}_w = \left\{ \mathcal{O}\left(\sum_{i=1}^{d+1} k_i D_i\right) : 0 \geq \sum \tilde{a}_i k_i > -\sum \tilde{a}_i \right\}.$$

*If  $q = r = 1$ , then the collection is complete.*

(ii) *Given a toric stack  $\mathcal{X} = \mathcal{X}_r$ , with MMP sequence*

$$(131) \quad \mathcal{X}_r \xrightarrow{f_r} \mathcal{X}_{r-1} \dashrightarrow \dots \xrightarrow{f_1} \mathcal{X}_0$$

*such that, to the birational map  $f_i$  corresponds to the wall  $w$  and has the associated toric stacks  $\mathcal{F}_i$ ,  $\mathcal{E}_i$  and  $\mathcal{B}_i$ . Then there is a semi-orthogonal decomposition*

$$(132) \quad D^b(\mathcal{X}) \simeq \langle \mathcal{S}_1, \dots, \mathcal{S}_r \rangle$$

*where  $\mathcal{S}_i$  admits a semi-orthogonal decomposition*

$$(133) \quad \mathcal{S}_i \simeq \langle j_*(\pi^*(D^b(\mathcal{B}_i)) \otimes \mathcal{L}) : \mathcal{L} \in \mathbf{E}_w \rangle.$$

*Proof.* For the most part, these statements are part of theorems 3.1, 4.3, 5.2 and 6.1 in [39]. The only additional point is the count of exceptional objects being  $\text{Vol}_0(A)$ . To see this, we observe that  $\phi : \mathbb{Z}^A \rightarrow \mathbb{Z}^d$  given by  $\phi(e_i) = \alpha_i$  has cokernel equal to  $\mathbb{Z}^d/\Lambda_w$  and rank 1 kernel. So the line bundles  $\mathcal{O}(\sum b_i D_i)$  form a subgroup of  $\text{Pic}(\mathcal{X})$  isomorphic to  $(\mathbb{Z}^d/\Lambda_w)^\vee \oplus \mathbb{Z}$ . Thus the line bundles  $\mathcal{O}(\sum k_i D_i)$  for  $0 \leq \sum k_i \tilde{a}_i > -\sum \tilde{a}_i$  can be counted up to equivalence as  $|\mathbb{Z}^d/\Lambda_w| \cdot (\sum \tilde{a}_i) = \text{Vol}_0(A)$ .  $\square$

One notational distinction worth noting is that what is called  $\mathcal{F}$  in [39], we refer to as  $\mathcal{B}$ .

We use this theorem to prove a more elementary result:

**Proposition 5.16.** *Let  $\mathcal{X}_\Sigma$  be a complete toric stack with simplicial stacky fan  $\Sigma$  in  $\mathbb{Z}^d$ . Then*

$$(134) \quad \text{rk}(K_0(D^b(\mathcal{X}_\Sigma))) = \text{Vol}(\Sigma) = \sum_{\sigma \in \Sigma(n)} \text{Mult}(\sigma)$$

*Proof.* We prove this by induction on dimension. Every stacky fan in  $\mathbb{Z}^1$  is given by two primitives  $\alpha_1, \alpha_2 \in \mathbb{Z}$  which yields the  $(2, 1)$  circuit  $A = \{\alpha_0, \alpha_1, \alpha_2\}$ . It is easy to compute that  $\text{Vol}_0(A) = |\alpha_1| + |\alpha_2|$  which equals the two quantities on the right in 134. By theorem 5.15 (i), we have that this is the number of exceptional objects in a complete exceptional collection, so the proposition holds for this case.

Now assume that the proposition holds for dimensions  $< d$  and all  $d$ -dimensional complete, simplicial stacky fans  $\tilde{\Sigma}$  with  $\text{Vol}(\tilde{\Sigma}) < V$  for some  $V \in \mathbb{N}$ . Let  $\Sigma$  be a  $d$ -dimensional complete, simplicial stacky fans  $\tilde{\Sigma}$  with  $\text{Vol}(\Sigma) = V$ . Let  $w$  be a wall in  $\Sigma$  and  $f : \Sigma \dashrightarrow \Sigma'$  the birational map associated to the circuit modification  $C(w)$  with signature  $(p, q; r)$ . Assume that the corresponding exceptional Mori fibration is  $\mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{B}$  and  $d_B = \dim(\mathcal{B})$ . Then we observe that theorem 5.15 (ii), the additivity of the rank of  $K_0$  relative to semi-orthogonal decompositions and the above assumptions imply

$$\begin{aligned} \text{rk}(K_0(D^b(\mathcal{X}_\Sigma))) &= \text{rk}(K_0(D^b(\mathcal{X}_{\Sigma'}))) + \text{Vol}_0(C(w)) \cdot \text{rk}(K_0(D^b(\mathcal{B}))), \\ &= \text{Vol}(\Sigma') + \text{Vol}_0(C(w)) \cdot \text{Vol}(\Sigma_B). \end{aligned}$$

Now, from the definition of  $\Sigma_B$ , we have that for every  $d$ -dimensional cone  $\tilde{\sigma} \in \bar{\Upsilon} \subset \Sigma$  containing  $\text{Cone}(\tau_j)$  as a subcone for some  $\tau_j \in \Upsilon$ , there is a unique  $\sigma \in \Sigma_B$  which is the image of  $\sigma' \in \Sigma$  for which  $\sigma' + \tau_j = \tilde{\sigma}$ . Now, the volume of the simplex  $t_\sigma$  associated to  $\tilde{\sigma}$  is  $\text{Vol}(\sigma) \cdot \text{Vol}(\tau_j)$  (note that we need to use  $\sigma$  in this formula instead of  $\sigma'$  in order to account for the lattice volume). The contribution to  $\text{Vol}(\Sigma)$  from  $\Upsilon$  is then  $\sum_{\tau_j \in \Upsilon, \sigma \in \Sigma_B} \text{Vol}(\tau_j) \cdot \text{Vol}(\sigma)$ .

The same statement holds for  $\bar{\Upsilon}^-$  so that the following formula holds for the difference

$$\begin{aligned} \text{Vol}(\Sigma) - \text{Vol}(\Sigma') &= \sum_{\tau_j \in \Upsilon, \sigma \in \Sigma_B} \text{Vol}(\tau_j) \cdot \text{Vol}(\sigma) - \sum_{\tau_i \in \bar{\Upsilon}^-, \sigma \in \Sigma_B} \text{Vol}(\tau_i) \cdot \text{Vol}(\sigma), \\ &= \sum_{\sigma \in \Sigma_B} \text{Vol}(\sigma) \left( \sum_{\tau_j \in \Upsilon} \text{Vol}(\tau_j) - \sum_{\tau_i \in \bar{\Upsilon}^-} \text{Vol}(\tau_i) \right), \\ &= \text{Vol}(\Sigma_B) \cdot \text{Vol}_0(\text{Core}(C(w))) = \text{Vol}(\Sigma_B) \cdot \text{Vol}_0(C(w)). \end{aligned}$$

But this implies that  $\text{rk}(K_0(D^b(\mathcal{X}_\Sigma))) = \text{Vol}(\Sigma) = V$  which proves the induction step.  $\square$



From this, we obtain an equivalence on the rank of the  $K$ -theory for the semi-orthogonal pieces arising from both the  $A$ -model and  $B$ -model categories.

**Corollary 5.17.** *Suppose  $\{[b_i, b_{i+1}]\}$  is the tight coherent subdivision of the vertex  $\psi \in \Sigma_{\rho_{\alpha_0}}(\Sigma(A))$  and  $\psi_t$  a regeneration of  $\psi$ . The semi-orthogonal decompositions  $Fuk^\rightarrow(\psi_t) = \langle \mathcal{T}_1, \dots, \mathcal{T}_r \rangle$  and  $D^b(\mathcal{X}_\Sigma) = \langle \mathcal{S}_1, \dots, \mathcal{S}_r \rangle$  have the property*

$$(135) \quad rk(K_0(\mathcal{T}_i)) = b_i - b_{i-1} = rk(K_0(\mathcal{S}_i)).$$

*Proof.* The statement for the equality  $rk(K_0(\mathcal{T}_i)) = b_i - b_{i-1}$  is referenced in the discussion following equation 116 that the multiplicity  $m_i$  of  $E_A$  equals change in the volume of the extended circuits supported along the circuit involved in the edge of the monotone path polytope. The equality for  $rk(K_0(\mathcal{S}_i))$  follows from the observation that  $b_i - b_{i-1} = \text{Vol}(\Sigma_i) - \text{Vol}(\Sigma_{i-1})$  obtained in the proof of the previous proposition.  $\square$

We infer from theorem 5.14 and proposition 5.17 the natural conjecture:

**Conjecture 5.18.** *Given any maximal degeneration  $\psi$  of an  $\{\alpha_0\}$ -sharpened pencil, let  $\psi_t$  be a regeneration of  $\psi$ . Let*

$$\begin{aligned} Fuk^\rightarrow(\psi_t) &= \langle \mathcal{T}_1, \dots, \mathcal{T}_r \rangle, \\ D^b(\mathcal{X}_\Sigma) &= \langle \mathcal{S}_1, \dots, \mathcal{S}_r \rangle \end{aligned}$$

*be the semi-orthogonal decompositions associated to  $\psi$  and its mirror sequence. Then there exists an equivalence of triangulated categories*

$$\Phi_\psi : Fuk^\rightarrow(\psi_t) \rightarrow D^b(\mathcal{X}_\Sigma)$$

*which restricts to equivalences  $\Psi_\psi : \mathcal{T}_i \rightarrow \mathcal{S}_i$  for all  $1 \leq i \leq r$ .*

In fact, a more detailed conjecture can easily be formulated about the equivalence of the categories  $\mathcal{T}_i$  and  $\mathcal{S}_i$  associated to degenerate circuits, but we will leave this to a later work.

Additional evidence for this conjecture comes from the case of  $A$  actually equaling a circuit, in which case this is simply the statement of homological mirror symmetry for a weighted projective stack. Certain classes of  $(2, 2)$  circuits were also examined in [40] where the equivalence of the circuit regeneration and the semi-orthogonal component associated to a blow-up was proved.

As a final remark, we point out that the edges of the monotone path polytope  $\Sigma_{\rho_{\alpha_0}}(\Sigma(A))$  correspond to minimal transitions between MMP sequences. They also correspond to certain two dimensional faces of  $\Sigma(A)$ . Restricting attention to those faces which have an edge on the minimum facet  $\rho_{\alpha_0} = 0$ , we obtain a transition between two Mori fiber spaces. Such moves, or links, have been well studied in a much more general context and their classification is referred to as the Sarkisov program. As an outgrowth of our perspective, one may pursue a complete structure theorem for all toric Sarkisov links.

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